Entailment Relations
Inference Systems
Confluent Relations

abstractions useful for logical systems

2006 / 6 / 19+21

Entailment Relations

$$A \vdash x$$

assumptions / axioms — consequence / conclusion

$A \subseteq \mathcal{X}$, $x \in \mathcal{X}$ statements

Definition

An entailment relation on a set $\mathcal{X}$ is a set $\vdash \; \subseteq \mathcal{P}\mathcal{X} \times \mathcal{X}$ such that for all $A, A', B \subseteq \mathcal{X}$:

1) $x \in A \Rightarrow A \vdash x$ Expansivity
2) $A \vdash x \wedge A \subseteq A' \Rightarrow A' \vdash x$ Monotonicity
3) $A \vdash B \wedge A \cup B \vdash x \Rightarrow A \vdash x$ Idempotence

where $A \vdash B \; :\Leftrightarrow \; \forall x \in B : A \vdash x$

Semantic entailment and deductive entailment are entailment relations on the set of all equations

Unify !

$A \vdash e$ defined by inference system $\Rightarrow$ $\vdash$ entailment relation

$A \models e \; :\Leftrightarrow \; \forall \mathfrak{A} : \mathfrak{A} \models A \Rightarrow \mathfrak{A} \models e$

$\mathfrak{A} \models A \; :\Leftrightarrow \; \forall e \in A : \mathfrak{A} \models e$ $\Rightarrow$ $\models$ entailment relation

Notations

$x_1, \dots, x_n \vdash x \quad :\Leftrightarrow \quad \{x_1, \dots, x_n\} \vdash x$

$A, x \vdash x' \quad :\Leftrightarrow \quad A \cup \{x\} \vdash x'$

$A, B \vdash x \quad :\Leftrightarrow \quad A \cup B \vdash x$

$A \vdash B \quad :\Leftrightarrow \quad \forall x \in B : A \vdash x$

$B \overset{A}{\vdash} B' \quad :\Leftrightarrow \quad A, B \vdash B'$

} reflexive and transitive

$A \dashv\vdash A' \quad :\Leftrightarrow \quad A \vdash A' \wedge A' \vdash A$

$B \overset{A}{\dashv\vdash} B' \quad :\Leftrightarrow \quad A, B \dashv\vdash A, B'$

} equivalence relations

Entailment Equivalence

$\boxed{A \dashv\vdash A' \Leftrightarrow \forall x \in \mathcal{X} : A \vdash x \Leftrightarrow A' \vdash x}$

If $A \dashv\vdash A'$, then

1) $A \cup B \dashv\vdash A' \cup B$
2) $A \vdash x \Leftrightarrow A' \vdash x$
3) $A \vdash B \Leftrightarrow A' \vdash B$
4) $B \vdash A \Leftrightarrow B \vdash A'$
5) $B \overset{A}{\dashv\vdash} B' \Leftrightarrow B \overset{A'}{\dashv\vdash} B'$

An entailment relation $\vdash$ is

- compact if
  $A \vdash x \Rightarrow \exists$ finite $A' \subseteq A : A' \vdash x$
- effective if $\mathcal{X}$ decidable and
  A semi-decidable $\Rightarrow \{x \mid A \vdash x\}$ semi-decidable

Deductive entailment is compact and effective

proof easy

Semantic entailment is neither compact nor effective

consider axiomatization of $\mathbb{N}$

Closure Operators

[Tarski 1930]

$[A] := \{x \mid A \vdash x\}$ closure of A

Closure operator $[\cdot] \in \mathcal{P}\mathcal{X} \to \mathcal{P}\mathcal{X}$
has nice algebraic properties:

1) $A \subseteq [A]$ expansivity
2) $A \subseteq B \Rightarrow [A] \subseteq [B]$ monotonicity
3) $[[A]] = [A]$ idempotence

entailment relation $\rightleftarrows$ closure operator

$A \vdash x \Leftrightarrow x \in [A]$

Inference Systems

- Describe possible inferences (i.e., inference steps)

$$\frac{x_1 \;\cdots\; x_n}{x} \qquad \begin{array}{l}\text{premises}\\ \text{conclusion}\end{array}$$

- Yield compact entailment relations

$A \vdash x \iff x$ can be obtained from the statements in A by finitely many inferences

Definition

An inference system on $\mathcal{X}$ is a set of pairs (P, x) such that P is a finite subset of $\mathcal{X}$ and $x \in \mathcal{X}$.

Derivations

Let S be an inference system on $\mathcal{X}$

A derivation of $x \in \mathcal{X}$ from $A \subseteq \mathcal{X}$ in S is a tuple $(x_1, \dots, x_n)$ such that

1) $x_n = x$
2) $\forall i \in \{1, \dots, n\} : x_i \in A \lor \exists P \subseteq \{x_1, \dots, x_{i-1}\} : (P, x_i) \in S$

$A \vdash_S x :\iff \exists$ derivation of x from A in S

$\boxed{\vdash_S \text{ is compact entailment relation}}$

$\boxed{\vdash_S \text{ is effective if } \mathcal{X} \text{ decidable and } S \text{ semi-decidable}}$

Closures

$S[A] := \{x \mid A \vdash_S x\}$ closure of A wrt S

A closed under S $:\iff \forall (P, x) \in S : P \subseteq A \Rightarrow x \in A$

A closed under S $\equiv$ A invariant for S

$\boxed{S[A] \text{ closed under } S}$

Closure Theorem

S inference system

$$\boxed{\left.\begin{array}{l} A \subseteq Q \\ Q \text{ closed under } S \end{array}\right\} \quad S[A] \subseteq Q}$$

Proof is straightforward

Encapsulates induction on length of derivation

Cor $\boxed{\begin{array}{l} S[A] \text{ is the least set that contains } A \\ \text{and is closed under } S \end{array}}$

Cor $\boxed{A \text{ closed under } S \iff A = S[A]}$

Closure Theorem is useful for proofs

Claim: $A \vdash e \implies \exists$ conversion proof of e from A

Proof: By closure theorem it suffices to show

1) $\forall e \in A$: $\exists$ conversion proof of e from A

2) $\forall (P, e) \in BED$:

$(\forall e \in P\ \exists$ conversion proof of e from $A)$

$\implies \exists$ conversion proof of e from A

$Q = \{ e \mid \exists \text{ conversion proof of } e \text{ from } A \}$

Inference systems are a tool for the recursive definition of sets

$\mathcal{X} = \mathbb{N}$

$S = \{ (\{x, y\}, x \cdot y) \mid x, y \in \mathbb{N} \}$ $\qquad \dfrac{x \quad y}{x \cdot y}$

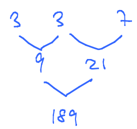
$S[A] \mathrel{\hat=}$ all numbers that can be obtained from A by multiplication

$S[\emptyset] = \emptyset$

$S[\{2\}] = \{ 2^n \mid n \geq 1 \}$

$S[\{3,7\}] = \{ 3, 7, 9, 21, 27, 49, \dots \}$

$= \{ 3^m \cdot 7^n \mid m + n \geq 1 \}$



Conversion Proofs

- alternative characterisation of deductive entailment
- based on replacement of equals with equals
- useful for hand calculations

$x = x \cdot 1$	Id	$x \cdot 1 = x$
$= x(x + \bar{x})$	Compl	$x + \bar{x} = 1$
$= xx + x\bar{x}$	Dist	$x(y+z) = xy + xz$
$= xx + 0$	Compl	$x\bar{x} = 0$
$= xx$	Id	$x + 0 = x$

conversion proof

conversion step

reduction step

Completion

β η instance

A

λ-reduction step

A-reduction step

Ref, Trans

Sym

CL, CR, ξ

Conversion Theorem

$$\boxed{A \vdash e \iff \exists \text{ conversion proof of } e \text{ from } A}$$

$(e_1, \dots, e_n)$

$e = e_n$

inferences according to

Ref, Sym, Trans, CL, CR, ξ, β, η

inference system

$(s_1, \dots, s_n)$

$e = (s_1, s_n)$

λ - conversions

A - conversions

rewrite system

Normal Form Theorem

[Church / Rosser / Tait]

$$\boxed{\emptyset \vdash s = t \iff s, t \text{ have the same } \lambda\text{-NF}}$$

similar result shown by Church/Rosser 1937

$s \xrightarrow{\lambda} t \;:\iff (s, t)$ is a λ-reduction step

$\xrightarrow{\lambda}$ is pronounced λ-reduction

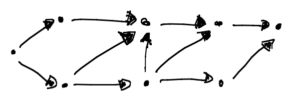
$$\boxed{\xrightarrow{\lambda} \text{ is confluent and terminating relation}}$$

termination shown by Tait 1967

Normal Form Theorem is straightforward consequence of Conversion Theorem and the confluence and termination of $\xrightarrow{\lambda}$

Confluent Relations

- Conversion proofs can be analyzed with binary relations on the set of terms
  replacement step $\rightsquigarrow (s, t)$
- $\rightarrow \subseteq \mathcal{X} \times \mathcal{X}$
- see $\rightarrow$ as graph
  (possibly infinite)



- useful abstractions evolved over a long time
  summarized by Huet 1980
- suggested reading: Baader/Nipkow

Reflexive Transitive Closure $\to^*$

$\to \subseteq \mathcal{X} \times \mathcal{X}$

$x \to^* y \;:\Longleftrightarrow\; \exists\, (x_1, \dots, x_n) \in \mathcal{X}^*:\; x = x_1 \to \cdots \to x_n = y$

$\exists$ path from x to y, y reachable from x

$\boxed{\to^* \text{ is reflexive, transitive and contains } \to}$

$\to^0 \;:=\; \{(x,x) \mid x \in \mathcal{X}\}$

$\to^n \;:=\; \to \circ \to^{n-1} \quad (n \geq 1)$

$\boxed{\to^* \;=\; \bigcup \{\to^n \mid n \in \mathbb{N}\}}$

Our definition of $\to^*$ is intuitively pleasing but inconvenient for proofs. The characterization with $\to^n$ solves the problem.

Connection with inference systems

$\boxed{\to \text{ reflexive} \iff \to \text{ closed under } \dfrac{}{(x,x)}\; x \in \mathcal{X}}$

$\boxed{\to \text{ transitive} \iff \to \text{ closed under } \dfrac{(x,y) \quad (y,z)}{(x,z)}\; x,y,z \in \mathcal{X}}$

$\boxed{\to^* \text{ is closure of } \to \text{ wrt } \dfrac{}{(x,x)}\; x \in \mathcal{X} \text{ and } \dfrac{(x,y) \quad (y,z)}{(x,z)}\; x,y,z \in \mathcal{X}}$

Characterization as inference closure

Proof

$\to^* \subseteq S[\to]$

We show $\forall n \forall x,y$: $x \to^n y \Rightarrow (x,y) \in S[\to]$ by induction on n.

Let $x \to^n y$

Case $n=0$. Then $x=y$ and hence $(x,y) \in S[\to]$

Case $n>0$. Then $x \to x' \to^{n-1} y$

$(x,x') \in S[\to]$

$(x',y) \in S[\to]$ induction

$(x,y) \in S[\to]$

$S[\to] \subseteq \to^*$

By Closure Theorem

1) $\to \;\subseteq\; \to^*$
2) $\to^*$ closed under reflexivity rule
3) $\to^*$ closed under transitivity rule

$\square$

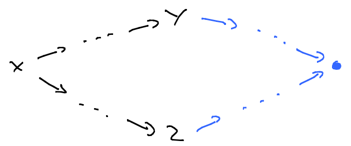
Corollary

$\boxed{\to^* \text{ is the least reflexive and transitive relation that contains } \to}$

$x \downarrow y \;:\Leftrightarrow\; \exists z:\; x \xrightarrow{*} z \,\wedge\, y \xrightarrow{*} z$

x, y joinable

$\rightarrow$ confluent $:\Leftrightarrow\; \forall x,y,z:\; x \xrightarrow{*} y \wedge x \xrightarrow{*} z \;\Rightarrow\; y \downarrow z$



$\rightarrow$ semi-confluent $:\Leftrightarrow\; \forall x,y,z:\; x \rightarrow y \wedge x \xrightarrow{*} z \;\Rightarrow\; y \downarrow z$

$\rightarrow$ locally confluent $:\Leftrightarrow\; \forall x,y,z:\; x \rightarrow y \wedge x \rightarrow z \;\Rightarrow\; y \downarrow z$

Conversion

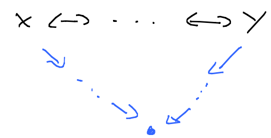
$x \leftrightarrow y \;:\Leftrightarrow\; x \rightarrow y \vee y \rightarrow x$

x, y one-step convertible

$x \xleftrightarrow{*} y \;\Leftrightarrow\; \exists (x_1, \ldots, x_n) \in X^*:\; x = x_1 \leftrightarrow \cdots \leftrightarrow x_n = y$

x, y convertible

$\rightarrow$ Church Rosser $:\Leftrightarrow\; \forall x,y:\; x \xleftrightarrow{*} y \Rightarrow x \downarrow y$



$\boxed{\rightarrow \text{ semi-confluent } \Rightarrow\; \rightarrow \text{ Church Rosser}}$

Proof Let $\rightarrow$ be semi-confluent.

To show: $\forall n\, \forall x,y:\; x \xleftrightarrow{n} y \Rightarrow x \downarrow y$.

By induction on n. Let $x \xleftrightarrow{n} y$.

Case $n=0$. Then $x=y$ and hence $x \downarrow y$.

Case $n>0$. Then $x \leftrightarrow x' \xleftrightarrow{n-1} y$

Case 1   Case 2

induction

semi-confluence □

Corollary

$\boxed{\rightarrow \text{ Church-Rosser} \Leftrightarrow \rightarrow \text{ confluent} \Leftrightarrow \rightarrow \text{ semi-confluent}}$

Termination

$\rightarrow$ terminating on x $:\Leftrightarrow$

$\neg \exists A \subseteq \mathcal{X}: \; x \in A \wedge \forall y \in A \; \exists z \in A: \; y \rightarrow z$

there exists no infinite path issuing from x

$\rightarrow$ terminating $:\Leftrightarrow$ $\rightarrow$ terminating for all $x \in \mathcal{X}$

Normal Forms

x NF $:\Leftrightarrow \neg \exists y: x \rightarrow y$ — terminal node

x NF for y $:\Leftrightarrow$ x NF and $y \rightarrow^* x$ — reachable terminal node

$\rightarrow$ terminating $\Rightarrow$ $\forall x \in \mathcal{X}$: x has NF

$\rightarrow$ confluent $\Rightarrow$ $\forall x \in \mathcal{X}$: x has at most one NF

If $\rightarrow$ terminating and confluent, then

1) $\forall x \in \mathcal{X}$: x has exactly one NF

2) $\forall x, y \in \mathcal{X}$: $\quad x \overset{*}{\leftrightarrow} y \Leftrightarrow x, y$ have the same NF

Proof: Exercise!

Induction Theorem

Often called well-founded induction
First studied by Emmy Noether 1882-1935

If $\rightarrow$ terminating, then

$\left(\forall x \in \mathcal{X}: \{y \mid x \rightarrow y\} \subseteq Q \Rightarrow x \in Q\right) \Rightarrow \mathcal{X} \subseteq Q$

Proof By contradiction

Let $\forall x \in \mathcal{X}: \{y \mid x \rightarrow y\} \subseteq Q \Rightarrow x \in Q$ (1)

$x \in \mathcal{X}, \; x \notin Q$ (2)

Then x NF w.l.o.g., $\rightarrow$ terminating, (1), (2)

$x \in Q$ (1)

$\square$

Newman's Lemma

Newman 1942

$\rightarrow$ terminatinating and locally confluent

$\Rightarrow$ $\rightarrow$ confluent

Proof of Huet 1980 is a very nice demonstration of general induction

Proof Let $\rightarrow$ be terminating and locally confluent

Show $\forall x \; \forall y, z \in \mathcal{X}: \; x \rightarrow^* y \wedge x \rightarrow^* z \Rightarrow y \downarrow z$

By induction on x wrt $\rightarrow$

$Q = \{x \mid \forall y, z \in \mathcal{X}: x \rightarrow^* y \wedge x \rightarrow^* z \Rightarrow y \downarrow z\}$

show: $\{y \mid x \rightarrow y\} \subseteq Q \Rightarrow x \in Q$

Let $x \rightarrow^* y \wedge x \rightarrow^* z$

Case $x = y \vee x = z$

Case $y \overset{*}{\leftarrow} x_1 \leftarrow x \rightarrow x_2 \rightarrow^* z$

$$
\begin{tikzcd}
y & x_1 \arrow[l, "*"'] \arrow[dr, "*"'] & x \arrow[l] \arrow[r] & x_2 \arrow[r, "*"] \arrow[dl, "*"] & z \arrow[ddll, "*"] \\
 & & x_3 \arrow[d, "*"] & & \\
 & & x_4 \arrow[dl, "*"] & & \\
 & x_5 & & &
\arrow[from=1-1, to=4-2, "*"']
\end{tikzcd}
$$

local confluence

induction for x_2

induction for x_1

$\square$