Higher-Order Propositional Logic

- Categorical axiomatization of Boolean operations
- Non-algebraic deductions

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Specification PL

defined constants (can be eliminated)

$0 : \mathbb{B}$		$1 = 0 \to 0$	D1
$\to : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$		$\neg x = x \to 0$	D¬
$0 \to x = 1$	I0	$x \vee y = (x \to y) \to y$	D∨
$1 \to x = x$	I1	$x \wedge y = \neg(\neg x \vee \neg y)$	D∧
$f0 \to f1 \to fx = 1$	BCA	$x \not\leftarrow y = \neg(y \to x)$	D↚
$x \vee y = y \vee x$	Com	$x \leftrightarrow y = (x \to y) \wedge (y \to x)$	D↔
		$x \not\leftrightarrow y = \neg(x \leftrightarrow y)$	D↮

$\mathcal{J}$: standard model of PL

$\boxed{\text{PL categorical}}$

- $\not\leftarrow$ serves as dual of $\to$
- Com semantically redundant
- only 4 essential axioms

Notation

Operator precedence

$\leftrightarrow$, $\not\leftrightarrow$

$\to$, $\not\leftarrow$

$\vee$

$\wedge$

$\doteq$, $\not\doteq$ notational variants of $\leftrightarrow$, $\not\leftrightarrow$; highest precedence

all operators associate to the right, e.g., $x \to y \to z \rightsquigarrow x \to (y \to z)$

$\bar{s} \rightsquigarrow \neg s$

Modus Ponens

$$\boxed{s \to t = 1,\ s = 1 \ \vdash_{PL}\ t = 1} \qquad \text{MP}$$

Use of BCA

$PL \vdash f_0 \to f_1 \to f\,x = 1$

$$\begin{tikzcd}[column sep=small] & f_x \arrow[dl, dash] \arrow[dr, dash] & \\ f_0 & & f_1 \end{tikzcd} \rightsquigarrow \begin{tikzcd}[column sep=small] & s \arrow[dl, dash] \arrow[dr, dash] & \\ s[x:=0] & & s[x:=1] \end{tikzcd}$$

$$\boxed{PL \vdash s[x:=0] \to s[x:=1] \to s = 1}$$

Proof. $s[x:=0] \to s[x:=1] \to s$

$= t\,0 \to t\,1 \to t\,x$ $t = \lambda x.s,\ \beta$

$= 1$ BCA □

$$\boxed{s[x:=0] = 1,\ s[x:=1] = 1 \ \overset{PL}{\vdash}\ s[x:=t] = 1}$$

0-1-Theorem

A term s is pure if it has the form

$s = 0 \mid 1 \mid x \mid \neg s \mid s \circ s$

$x : B$ $\circ = \to \mid \not\leftrightarrow \mid \land \mid \lor \mid \leftrightarrow$

pure term ≅ Boolean term

$$\boxed{s \text{ pure and closed} \implies PL \vdash s = 0 \ \lor\ PL \vdash s = 1}$$

Proof. By induction on $|s|$.

$$\boxed{s \text{ pure and closed} \implies (PL \models s = 1 \iff PL \vdash s = 1)}$$

Corollary to the statement above.

Completeness for pure terms

$$\boxed{s \text{ pure} \implies (\mathcal{I} \models s = 1 \iff PL \vdash s = 1)}$$

Proof. Let s be pure.

"$\Leftarrow$" Let $PL \vdash s = 1$. Then $PL \models s = 1$ by Soundness and $\mathcal{I} \models s = 1$ by $\mathcal{I} \models PL$.

"$\Rightarrow$" Show $\forall s$ pure: $\mathcal{I} \models s = 1 \Rightarrow PL \vdash s = 1$
by induction on number n of variables in s.
Let s be pure and $\mathcal{I} \models s = 1$. (1)
Case $n = 0$. By Soundness $PL \nvdash s = 0$. Hence $PL \vdash s = 1$ 0-1-Theo.
Case $n > 0$. Let $x \in \mathcal{N}s$. Then
$\mathcal{I} \models s[x:=0] = 1,\ \mathcal{I} \models s[x:=1] = 1$ (1)
$PL \vdash s[x:=0] = 1,\ PL \vdash s[x:=1] = 1$ Induction
$PL \vdash s = 1$ BCA □

$$\boxed{s \land t = 1 \ \overset{PL}{\dashv\vdash}\ s = 1,\ t = 1}$$

And

Proof "$\dashv$" 0-1-Theo

"$\vdash$"
$PL \vdash x \land y \to x = 1$ Completeness
$PL \vdash x \land y \to y = 1$ Completeness
$s \land t = 1 \overset{PL}{\vdash} s = 1, t = 1$ Generativity, MP □

Eq

$$\boxed{s = t \;\overset{PL}{\dashv\vdash}\; s \doteq t = 1}$$

This is the only result for which Com is needed, and Com is included to get this result

Proof $\vdash$ Follows with $PL \vdash x \doteq x = 1$

$\dashv$ $s \doteq t = 1 \;\overset{PL}{\vdash}\; (s \to t) \wedge (t \to s) = 1$ by D⇔

$\overset{PL}{\vdash}\; s \to t = 1,\; t \to s = 1$ by And (1)

$\overset{PL}{\vdash}\; s = t$ by (2)

(2)

$$\begin{aligned}
s &= 1 \to s && \text{I1}\\
&= (t \to s) \to s && (1)\\
&= t \vee s && \text{D}\vee\\
&= s \vee t && \text{Com}\\
&= (s \to t) \to t && \text{D}\vee\\
&= 1 \to t && (1)\\
&= t && \text{I1} \quad \square
\end{aligned}$$

Completeness for pure equations

$$\boxed{e \text{ pure} \Rightarrow (\mathcal{J} \models e \iff PL \vdash e)}$$

- $s = t$ pure if s, t pure
- pure equation ≙ Boolean equation

Proof Compl pure terms, Eq, Soundness □

Cor $\boxed{PL \vdash BA}$ $\wedge = \cdot\,,\ \vee = +$

Can now reuse deductive results for BA

Duality

$\delta 0 = 1$	$\delta 1 = 0$
$\delta\wedge = \vee$	$\delta\vee = \wedge$
$\delta\!\to\, = \not\leftarrow$	$\delta\!\not\leftarrow\, = \to$
$\delta\!\Leftrightarrow\, = \not\Leftrightarrow$	$\delta\!\not\Leftrightarrow\, = \Leftrightarrow$

$\boxed{\delta(\delta s) = s}$ $\boxed{PL \vdash \delta(PL)}$

$\boxed{PL \vdash e \iff PL \vdash \delta e}$

Tautologies and Practical Advice

$\text{Taut} := \{ e \text{ pure} \mid \mathcal{J} \models e \}$

$\boxed{e \in \text{Taut} \iff e \text{ pure} \wedge PL \vdash e}$

To prove $e \in \text{Taut}$, BA-style reasoning works well

To prove $PL \vdash e$, a combination of
- Eq
- BCA
- Taut-conversions

may work well

Boolean Replacement

$$\boxed{PL \vdash x \doteq y \to fx \;=\; x \doteq y \to fy} \qquad \textbf{BRep}$$

- Internal formulation of "Replacement of equals with equals"
- Useful for conversion proofs
- Not pure

Proof of BRep

Claim $PL \vdash x \doteq y \to fx \;=\; x \doteq y \to fy$

Proof Eq, BCA, Taut

Let $A = x \doteq y \to fx \Leftrightarrow x \doteq y \to fy$

Show $PL \vdash A = 1$ with BCA and Taut (+MP, Gen)

$$\begin{array}{cccc} & A & & \\ A[x:=0] & & A[x:=1] & \\ A[x:=0][y:=0] & A[x:=0][y:=1] & A[x:=1][y:=0] & A[x:=1][y:=1] \\ & 0 \doteq 1 \to f0 \Leftrightarrow 0 \doteq 1 \to f1 & & 1 \doteq 1 \to f1 \Leftrightarrow 1 \doteq 1 \to f1 \\ & 0 \doteq 1 \to x \Leftrightarrow 0 \doteq 1 \to y & & x \Leftrightarrow x \end{array}$$

instances of tautologies

$\square$

Variants of BRep

$$\boxed{PL \vdash x \doteq y \wedge fx = x \doteq y \wedge fy}$$

$$\boxed{PL \vdash x \doteq y \to fx \to fy = 1}$$

Generalized BCA (Proof Technique)

Show $PL \vdash A = 1$ ($PL \vdash s$ with Eq)

by iterated case analysis with the tautologies

$x = (y \doteq 0 \to x) \wedge (y \doteq 1 \to x)$ GBCA

$x \to y \wedge z = (x \to y) \wedge (x \to z)$

and BRep (exploiting the premises introduced by GBCA)

and Taut-conversion steps

$$\begin{array}{ccc} & A & \\ t{=}0 \to A & & t{=}1 \to A \\ \wedge & & \wedge \end{array}$$

goal

subgoals

- simplify goals with BRep and Taut-Conversion
- results in a conversion proof

Example

Claim $PL \vdash f(f(f1)) \to f1 \doteq 1$

Proof

$$\dfrac{\dfrac{f\underline{0} \to 0 \quad 0 \to 0 \quad 1}{} \quad \dfrac{\underline{f1} \to 0 \quad 0 \to 0 \quad 1}{}}{f(f\underline{0}) \to 0} \qquad f(f1) \to 1 \quad 1$$

$f1 \doteq 0$	$f(f\underline{0}) \to 0$		$f(f1) \to 1$	$f1 \doteq 1$
			1	Taut $(x \to 1 = 1)$
$f0 \doteq 0$	$f\underline{0} \to 0$	$\underline{f1} \to 0$	$f0 \doteq 1$	
$f0 \doteq 0$	$0 \to 0$	$0 \to 0$	$f1 \doteq 0$	
Taut	1	1	Taut	

Top: $f(f(\underline{f1})) \to \underline{f1}$, split on $f1$; left branch split on $f0$.

red underline $\underline{\ }$ $\hat{=}$ BRep-Conversion

Deductivity

- Tool for proving implications, proof transformer
- Relates external $\vdash$ to internal $\to$
- External version of BRep

$$\boxed{s \text{ closed} \wedge A, s \doteq 1 \vdash^{PL} t \doteq 1 \Rightarrow A \vdash^{PL} s \to t \doteq 1}$$

Proof. Let s closed, $A, s \doteq 1 \vdash^{PL} t \doteq 1$.

$\exists$ conversion proof of $t \doteq 1$ from $A, s \doteq 1, PL$

$\exists$ conversion proof of $s \doteq 1 \to t = s \doteq 1 \to 1$ from A, PL

- s closed $\Rightarrow$ s is the only instance of s
- Replacement $s \leftrightarrow 1$ justified by BRep

- $BA \vdash s \doteq 1 \to t = s \to t$
- $BA \vdash s \doteq 1 \to 1 = 1$ $\square$

Consistency and Satisfiability

A consistent $:\Leftrightarrow \neg \exists x, y: A \vdash x = y \wedge x \neq y$

A satisfiable $:\Leftrightarrow A$ has a proper model

$$\boxed{A \text{ satisfiable} \Rightarrow A \text{ consistent}}$$

If s closed, then

$A \vdash^{PL} s \doteq 1 \Leftrightarrow A, \bar{s} \doteq 1 \vdash^{PL} 0 \doteq 1$

$\Leftrightarrow PL \cup A \cup \{\bar{s} \doteq 1\}$ inconsistent

Follows with Deductivity

Variables versus parameters

$A \vdash e \;,\; A \models e$

- Can always assume A closed
- Can always assume e closed*

$e = (x \doteq y \to fx = x \doteq y \to fy)$

$\theta = \{x := a, y := b\}$

$A \vdash \theta e$ by Generativity

$\psi = \{a := x, b := y\}$

$\psi A \vdash \psi(\theta e)$ by Stability

$A \vdash e$ since $\psi A = A$ and $\psi(\theta e) = e$

* provided A frugal, i.e., for every type there are infinitely many constants not used in A

Propositional Completeness

Plain terms, equations: like pure but parameters in place of variables: $s = a \mid \neg s \mid s \circ s$ where $a : \mathbb{B}$

$$\boxed{A, e \text{ plain} \Rightarrow \left(A \models^{PL} e \Leftrightarrow A \vdash^{PL} e\right)}$$

suffices to show

$$\boxed{A \text{ plain} \wedge PL \cup A \text{ consistent} \Rightarrow PL \cup A \text{ satisfiable}}$$

Proof can be based on ACCs, see [Fitting], [Andrews]

For finite A, the above follows from previous results.

Propositional Compactness

$$\boxed{\begin{array}{l}\text{If } A \text{ plain, then} \\ PL \cup A \text{ satisfiable} \Leftrightarrow \forall A' \subseteq A \text{ finite: } PL \cup A' \text{ satisfiable}\end{array}}$$

straightforward consequence of Prop. Completeness