# Boolean Algebra

BA axiomatizes

1) logical operations $\wedge, \vee, \neg$ (Boole 1854)
2) set operations $\cap, \cup$, complement

- first instance of abstract algebra

# Tautology Theorem

$\boxed{\begin{array}{l} \text{For every proper model } \mathcal{L} \text{ of BA and} \\ \text{for every B. equation } e \\ \quad BA \vdash e \overset{1}{\Leftrightarrow} BA \models e \overset{2}{\Leftrightarrow} \mathcal{L} \models e \end{array}}$

- Equivalence 1 may be obtained from a completeness theorem for algebraic specifications (Birkhof 1930)
- Equivalence 2 is a special property of BA

  Properties of logical operations $\hat{=}$ properties of set operations

# Quiz

Find an equation $e$ s.t.

1) $\mathcal{J} \models e$
2) $BA \not\models e$

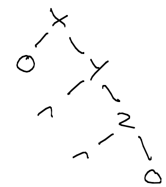
# Prime Trees

$\boxed{\forall s \in BT \ \exists^{1} t \in PT : \ BA \vdash s = t}$

$\boxed{\begin{array}{l} \text{For all B. terms } s, t \\ BA \vdash s = t \Leftrightarrow \pi s = \pi t \end{array}}$

$PT \subset DT \subset BT$

ordered, reduced



## Conditionals

$(s, t_0, t_1) \rightsquigarrow \bar{s}\cdot t_0 + s\cdot t_1$

if $s=0$ then $t_0$ else $t_1$

Surprise: properties of 2-valued conditionals are deducible in BA

$(x, y, y) = y$

$(x, y, z) \circ u = (x, y\circ u, z\circ u)$

$(x, y, z) \circ (x, y', z') = (x, y\circ y', z\circ z')$

## 0-1 Theorem

$$\boxed{s \in BT \text{ closed} \implies BA \vdash s = 0 \ \lor\ BA \vdash s = 1}$$

## Expansion Theorem

$$\boxed{s, x \in BT \implies BA \vdash s = (x, s[x:=0], s[x:=1])}$$

Suffices to show

1) $BA \vdash x\cdot s = x\cdot(s[x:=1])$

2) $BA \vdash \bar{x}\cdot s = \bar{x}\cdot(s[x:=0])$

Proof by induction on $s$, using the following tautologies

$x\cdot x = x\cdot 1$ $\qquad x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z)$

$x\cdot\bar{y} = x\cdot\overline{xy}$ $\qquad x\cdot(y+z) = x\cdot y + x\cdot z$

## Canonicity Lemma

$\mathfrak{B}$ proper model of BA
$s, t$ different prime trees
$\implies \exists \mathfrak{M}: \mathfrak{B} \le \mathfrak{M} \land \hat{\mathfrak{M}}s \ne \hat{\mathfrak{M}}t$

Proof is somewhat tricky

requires notion of

significant variables $\quad SV_{\mathfrak{B}}\, s$

# Significant Variables

$\mathcal{L}$ proper model of BA

$SV_{\mathcal{L}}s := \{ x \mid \tau x = B \wedge \exists \mathcal{M}: \mathcal{L} \leq \mathcal{M} \wedge \exists v \in \mathcal{L}B: \hat{\mathcal{M}}s \neq \hat{\mathcal{M}}_{x,v} s \}$

F0 $\boxed{BA \models s = t \Rightarrow SV_{\mathcal{L}}s = SV_{\mathcal{L}}t}$

F1 $\boxed{SV_{\mathcal{L}}s \subseteq \mathcal{N}s}$ Coincidence

# More Facts

$\mathcal{L}$ proper model of BA, $\mathcal{L} \leq \mathcal{M}$

F2
$$\boxed{\begin{array}{l} \mathcal{M}x = \mathcal{L}0 \Rightarrow \mathcal{M}(x,s,t) = \mathcal{M}s \\ \mathcal{M}x = \mathcal{L}1 \Rightarrow \mathcal{M}(x,s,t) = \mathcal{M}t \end{array}}$$

F3 $\boxed{\mathcal{M}s_0 \neq \mathcal{M}s_1 \wedge x \notin \mathcal{N}s_0 \cup \mathcal{N}s_1 \Rightarrow x \in SV_{\mathcal{L}}(x, s_0, s_1)}$ F2

F4
$$\boxed{\begin{array}{l} \mathcal{M}s_0 \neq \mathcal{M}t_0 \wedge x \notin \mathcal{N}s_0 \cup \mathcal{N}t_0 \\ \quad \Rightarrow \exists \mathcal{M}': \mathcal{L} \leq \mathcal{M}' \wedge \hat{\mathcal{M}}'(x, s_0, s_1) \neq \hat{\mathcal{M}}'(x, t_0, t_1) \end{array}}$$
$\mathcal{M}' = \mathcal{M}_{x,\mathcal{L}0}$ F2

$$\boxed{\begin{array}{l} \mathcal{L} \text{ proper model of BA and } s,t \in PT. \text{ Then} \\ 1)\ x \in \mathcal{N}s \iff x \in SV_{\mathcal{L}}s \\ 2)\ s \neq t \Rightarrow \exists \mathcal{M}: \mathcal{L} \leq \mathcal{M} \wedge \hat{\mathcal{M}}s \neq \hat{\mathcal{M}}t \end{array}}$$

Proof. By ind on $|s| + |t|$, both claims together

1) "$\Leftarrow$" by Coincidence. "$\Rightarrow$" by case analysis.

Case $s \in \{0,1\}$. ✓

Case $s = (x, s_0, s_1)$. Follows with IH (2) and F3

2) Case $s, t \in \{0,1\}$. ✓

Case "The root variable of $s$ doesn't occur in $t$ or vice versa"
Follows with (1) and F0

Case "$s$ and $t$ have the same root variable"
Follows with IH (2) and F4 □

# Modelling with Boolean Equations

- Will consider an application
- Switch to ordinary math mode
- Boolean will mean 2-valued, i.e., $\mathbb{B} = \{0,1\}$

## Diet Rules of an Old Gentleman

1) If you don't drink beer, always eat fish

2) If you have both beer and fish, don't eat ice cream

3) If you don't drink beer or eat ice cream, don't have fish

## Model the Diet Rules

with 3 Boolean observables

B: meal includes beer

F: meal includes fish

I: meal includes ice cream

and one equation per rule

$$\boxed{\begin{aligned} \overline{B} \to F &= 1 \\ B \wedge F \to \overline{I} &= 1 \\ \overline{B} \vee I \to \overline{F} &= 1 \end{aligned}}$$

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## When the theory pays off

What you have learned so far in this course doesn't help with the modelling but once you have the model it pays off:

1) Are there meals that satisfy the rules?
Is the equation system solvable?

2) Are there other (simpler) diet rules that license exactly the same meals?
Are there equivalent equation systems?

## Definition of Solutions

$\mathfrak{A}$ : structure

$\mathcal{U} \subseteq Var \cup Con$, $\mathcal{U} \cap \Sigma_{\mathfrak{A}} = \emptyset$ Unknowns

$Sol_{\mathfrak{A},\mathcal{U}}\, E := \{\sigma \mid \exists \mathfrak{I}: \mathfrak{A} \leq \mathfrak{I} \wedge \mathfrak{I} \models E \wedge \sigma \leq \mathfrak{I} \wedge Dom\,\sigma = \mathcal{U}\}$

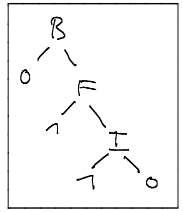
# Equation System ⟶ Boolean Term

Formalize the equation system $E$ as a B.term $s$

$$(\bar{B} \to F)\,(B \wedge F \to \bar{I})\,(\bar{B} \vee I \to \bar{F})$$

- Solutions $E$ = Solutions $s = 1$
- $E$ solvable $\iff$ $\mathcal{T} \not\models s = 0$
- $\mathcal{T} \models s = t \implies$ Solutions $s = 1$ = Solutions $t = 1$
- $\Pi s$ can serve as solved form for $E$

# Solving $E$ $\hat{=}$ Simplifying $s$

$\Pi s =$ 

$$
\begin{aligned}
s &= (\bar{B} \to F)\,(B \wedge F \to \bar{I})\,(\bar{B} \vee I \to \bar{F}) \\
&\equiv (\bar{\bar{B}} + F)\,(\overline{B \cdot F} + \bar{I})\,(\overline{\bar{B} + I} + \bar{F}) \\
&\equiv (B + F)\,(\bar{B} + \bar{F} + \bar{I})\,(B \cdot \bar{I} + \bar{F}) \\
&\equiv (B + F)\,(\bar{B} + \bar{F} + \bar{I})\,(B + \bar{F})\,(\bar{I} + \bar{F}) \\
&\equiv (\quad)\,(\quad)\,(\quad)\,(\quad)\,B && \text{Resolution} \\
&\equiv (\bar{I} + \bar{F})\,B && \text{Absorption}
\end{aligned}
$$