Predicate Logic with Choice

- PL + axiomatization of choice operator
- $CL \vdash QL \vdash PL \vdash BA$
- Skolem quantifier elimination

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Specification CL

Extends	PL		
Constants	$C_T : (T \to B) \to T$		choice
Axioms	$fx \to f(Cf) = 1$	CI	
Derived Constants	$\bar{C}f = C(\lambda x.\overline{fx})$	$D\bar{C}$	dual choice
	$\exists f = f(Cf)$	$D\exists$	
	$\forall f = f(\bar{C}f)$	$D\forall$	

C not uniquely determined, 4 possibilities for B

$\boxed{CL \vdash QL}$

Can reuse proof techniques for QL

Claim $CL \vdash \forall I$

Proof

$$\begin{aligned} \forall f \to fx &= \overline{fx} \to \overline{\forall f} && \text{Taut} \\ &= \overline{fx} \to \overline{f(C(\lambda x.\overline{fx}))} && D\forall,\ D\bar{C} \\ &= px \to p(Cp) && p = \lambda x.\overline{fx},\ B \\ &= 1 && CI \end{aligned}$$

$\square$

Exercise Prove $QL \vdash \exists 0, \exists I, \forall 1$.

Duality

$\delta 0 = 1$	$\delta 1 = 0$
$\delta \wedge = \vee$	$\delta \vee = \wedge$
$\delta \to = \not\leftarrow$	$\delta \not\leftarrow = \to$
$\delta \leftrightarrow = \not\leftrightarrow$	$\delta \not\leftrightarrow = \leftrightarrow$
$\delta C = \bar{C}$	$\delta \bar{C} = C$
$\delta \forall_T = \exists_T$	$\delta \exists_T = \forall_T$

$\boxed{\delta(\delta s) = s}$

$\boxed{CL \vdash \delta(CL)}$

$\boxed{CL \vdash e \Leftrightarrow CL \vdash \delta e}$

Claim $CL \vdash fx \not\to f(\bar{C} f) = 0$ dual of CI

Proof
$$fx \not\to f(\bar{C} f)$$
$$= \overline{\overline{fx} \to \overline{f(\bar{C} f)}} \qquad D\not\to,\ \text{Taut}$$
$$= \overline{px \to p(Cp)} \qquad p = \lambda x.\overline{fx},\ D\bar{C},\ \beta$$
$$= \bar{1} \qquad CI$$
$$= 0 \qquad \text{Taut} \qquad \square$$

Skolem's Law

$$\boxed{CL \vdash \forall x \exists y.\ fxy = \exists g \forall x.\ fx(gx)} \qquad \text{Sko}$$

$QL \models \text{Sko}$

$QL \not\vdash \text{Sko}$ (difficult to show)

Proof of Skolem's Law

$\exists\not\to$, $D\Leftrightarrow$, And

$$(\forall x \underbrace{\exists y.\ fxy}_{\exists(fx)}) \to \exists g \forall x.\ fx(gx) \qquad \eta$$
$$\exists g.\ (\forall x.\ fx(C(fx))) \to \forall x.\ fx.(gx) \qquad D\exists,\ \text{Pull}\to$$
$$\text{Gen}\exists:\ g := \lambda x.\ C(fx)$$
$$\beta$$
$$1 \qquad \text{Taut}$$

$$(\exists g \forall x.\ fx(gx)) \to \forall x \exists y.\ fxy$$
$$\exists z \exists y.\ fz(gz) \to fxy \qquad \text{Pull}\to,\ \text{Gen}\forall$$
$$fx(gx) \to fx(gx) \qquad \text{Gen}\exists:\ z := x,\ y := gx$$
$$1 \qquad \text{Taut}$$

Prenex Form

$\underbrace{Q_1 x_1 \ldots Q_n x_n}_{\text{quantifier prefix}}.\ \underbrace{p}_{\text{body}}$ where $n \geq 0$ and $\exists, \forall, C \notin \mathcal{N}p$ and $p : \mathbb{B}$

Skolem Form

$\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n .\ p$ where $m, n \geq 0$ and $\exists, \forall, C \notin \mathcal{N}p$ and $p : \mathbb{B}$

> Equivalent Skolem form can be obtained from prenex form with Skolem's law

Prenex form often can be obtained with pull laws and dM

Skolem Quantifier Elimination (Skolemization) [1928]

Let A be in prenex form and frugal.
Then there exists for each s an A' such that

1) $\forall, \exists, C \notin \mathcal{N}(A')$
2) $A \vdash^{CL} s = 1 \iff A' \vdash^{CL} s = 1$

A first-order $\rightsquigarrow$ A' algebraic

Lemma

If $\exists x.s$ closed and a does not appear in A, s and $t=1$, then

$$A, \exists x.s = 1 \vdash^{QL} t = 1 \iff A, s[x:=a] = 1 \vdash^{QL} t = 1$$

Outermost existential quantifiers can be eliminated by introducing so-called Skolem constants

$$A, \exists x.s = 1 \vdash^{QL} t = 1$$

$\iff A \vdash^{QL} (\exists x.s) \to t = 1$ Deductivity, $\exists x.s$ closed

$\iff A \vdash^{QL} \forall x.\, s \to t = 1$ Pull $\to$

$\iff A \vdash^{QL} s \to t = 1$ Gen $\forall$

$\iff A \vdash^{QL} s[x:=a] \to t = 1$ Stability, $a \notin \mathcal{N}(A, s, t, 1)$, $x \notin \mathcal{W}t$

$\iff A, s[x:=a] = 1 \vdash t = 1$ Deductivity, $s[x:=a]$ closed

Example

$A = \{ \forall x.\, fx \to \exists f \wedge gx = 1 \}$ f, g constants

$\forall x.\, fx \to \exists f \wedge gx$

$= \forall x \exists y.\; fx \to fy \wedge gx$ Pull

$= \exists z \forall x.\; fx \to f(zx) \wedge gx$ Skolem

$A' = \{ fx \to f(ax) \wedge gx = 1 \}$ a new constant