Predicate Logic with Identity

- $QL' = QL + \{\doteq, \not\doteq\}$ as derived constants
- $QL'' = QL' +$ Extensionality

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$QL' = QL + \{\doteq, \not\doteq\}$

$x \doteq y = \forall f.\ fx \to fy$ $D\doteq$ Leibniz

$x \not\doteq y = \neg(x \doteq y)$ $D\not\doteq$

Duality of QL preserved with

$\delta(\doteq) = (\not\doteq)$ $\delta(\not\doteq) = (\doteq)$

$QL \vdash \forall f.\ fx \to fy = \forall f.\ fx \leftrightarrow fy$

Basic Identity Laws

The followings equations are deducible in QL'

Ref	$x \doteq x = 1$	
Sym	$x \doteq y = y \doteq x$	
Trans	$x \doteq y \to y \doteq z \to x \doteq z = 1$	
CR	$x \doteq y \to gx \doteq gy = 1$	
CL	$g \doteq h \to gx \doteq hx = 1$	consequence of CR
Rep	$x \doteq y \to fx = x \doteq y \to fy$	generalizes BRep
$D\doteq'$	$x \doteq y = \forall f.\ fx \leftrightarrow fy$	

Claim $QL' \vdash x \doteq y = y \doteq x$

Proof Eq, $D\leftrightarrow$, And

	$x \doteq y \to y \doteq x$	
$=$	$(\forall f.\ fx \to fy) \to \forall g.\ gy \to gx$	$D\doteq$
$\vdash$	$\exists f.\ (fx \to fy) \to gy \to gx$	Pull, Gen$\forall$
$=$	$\exists f.\ (fx \to fy) \to \overline{gx} \to \overline{gy}$	Taut (Contraposition)
$\dashv$	$(\overline{gx} \to \overline{gy}) \to \overline{gx} \to \overline{gy}$	Gen$\exists$: $f = \lambda x.\overline{gx}$, β
$=$	1	Taut $\square$

Rep provides for capture-free replacement

$$\boxed{QL' \vdash \quad s_1 \doteq s_2 \to t[x := s_1] \;=\; s_1 \doteq s_2 \to t[x := s_2]}$$

$$(\lambda x.t)\, s_1 \qquad\qquad (\lambda x.t)\, s_2$$

Extensionality

Ext $\quad (\forall x.\ fx \doteq gx) \to f \doteq g \;=\; 1$

$\boxed{QL' \models Ext}$ obvious

$\boxed{QL' \not\vdash Ext}$ difficult (uses non-standard interpretations)

$\boxed{CL' \not\vdash Ext}$ conjecture

$QL'' := QL' \cup Ext \qquad CL'' := CL' \cup Ext$

Equivalent variants of Ext

$$\boxed{\forall x.\ fx \doteq gx \;=\; f \doteq g \quad \vdash\!\!\frac{QL'}{}\!\!\dashv \quad Ext}$$

$$\boxed{\forall x.\ s \doteq t \;=\; (\lambda x.s) \doteq (\lambda x.t) \quad \vdash\!\!\frac{QL'}{}\!\!\dashv \quad Ext}$$

Ext provides for replacement with legal capture

$$\boxed{QL'' \vdash (\forall x.\ s \doteq t) \to f(\lambda x.\ g\, s) \;\doteq\; f(\lambda x.\ g\, t) \;=\; 1}$$

Ext $\qquad$ CR $\qquad$ β

$(\lambda x.s) \doteq (\lambda x.t) \qquad f(\lambda x.\ g((\lambda x.s)\, x))$

$(\lambda h.\ f(\lambda x.\ g(h x)))\ (\lambda x.s)$

Relationship to $\xi \quad \dfrac{s = t}{\lambda x.s = \lambda x.t}$ at meta level