First-Order Predicate Logic (FOPL)

- Fragment of predicate logic with complete deduction
- Thoroughly studied
- Most textbook present first-order predicate logic only

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Formulas

Formula: term of type B

Atomic formula: formula such that no proper subterm is a formula

S set of formulas

$S \vdash^{A} s \;:\Leftrightarrow\; \{s=1 \mid s \in S\} \vdash^{A} s=1$

$S \models^{A} s \;:\Leftrightarrow\; \{s=1 \mid s \in S\} \models^{A} s=1$

S A-consistent $:\Leftrightarrow\; S \nvdash^{A} 0$

S A-satisfiable $:\Leftrightarrow\; \{s=1 \mid s \in S\} \cup A$ has a model $\mathfrak{A}$ s.t. $\mathfrak{A}0 \neq \mathfrak{A}1$

First-Order Formulas (fof's)

α atomic and algebraic formula

$s = \alpha \mid \neg s \mid s \wedge s \mid s \vee s \mid \forall x.s \mid \exists x.s$

where τx non-functional

normal fof: closed fof where $\neg$ is only applied to atomic formulas

Important Results for FOPL

- If S, s first-order, then
  $S \models^{QL'} s \iff S \vdash^{QL'} s$

  Completeness
  Gödel 1929

- $\{s \mid \emptyset \vdash^{NL} s,\ s \text{ fof licensed by } NL\}$
  is not semi-decidable where
  NL is axiomatization of $\mathbb{B}, \mathbb{N}$
  with $\wedge, \neg, \forall_{\mathbb{N}}, \underline{0}, 1, +, \cdot, \doteq_{\mathbb{N}}, \leq$

  Incompleteness
  Gödel 1931

- $\{s \mid \emptyset \models^{QL'} s,\ s \text{ fof}\}$ undecidable

  Undecidability
  Church 1936

Model Existence Theorem (MET)

$\boxed{\text{If } S \text{ set of normal fofs, then } S \text{ QL'-consistent} \Rightarrow S \text{ QL'-satisfiable}}$

Corollary (Completeness)

$\boxed{\text{If } S, s \text{ are first-order, then } S \models_{QL'} s \Leftrightarrow S \vdash_{QL'} s}$

Follows with deductivity
$S \vdash_{QL'} s \Leftrightarrow S, \bar{s} \vdash_{QL'} 0$

- First MET shown by Kurt Gödel, 1929, in his Doctoral Thesis, University of Vienna, "Über die Vollständigkeit des Logikkalküls"
- Modern proofs
  - Henkin 1949
  - Smullyan 1963
- Textbooks
  - Andrews 2002
  - Fitting 1996

Proof of MET, Ideas

Idea: if S unsat, a conflict will appear after finitely many consequences are added

- Suffices to consider only normal fofs
- Unsatisfiability can be shown with finitely many steps, i.e., a refutation
- Leads to a proof method known as semantic tableau

1) Identify obvious conflicts such that
   S contains a conflict $\Rightarrow$ S unsat

2) Add consequences to S such that
   a) S unsat $\Leftrightarrow$ $S \cup \{s\}$ unsat — linear consequence
   b) S unsat $\Leftrightarrow$ $S \cup \{s_1\}$ unsat $\wedge$ $S \cup \{s_2\}$ unsat — binary consequence leads to branching

3) Unsatisfiability of S can be made explicit (i.e., through conflict) by adding finitely many consequences

Refutation of $(a \vee a) \wedge (\bar{a} \vee \bar{a})$

partial refutations are called tableaus

1	$(a \vee a) \wedge (\bar{a} \vee \bar{a})$				
2	$a \vee a$				(1)
3	$\bar{a} \vee \bar{a}$				(1)
4	a		a		(2)
5	$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{a}$	(3)

$\{(a \vee a) \wedge (\bar{a} \vee \bar{a}),\ a \vee a,\ \bar{a} \vee \bar{a},\ a,\ \bar{a}\}$

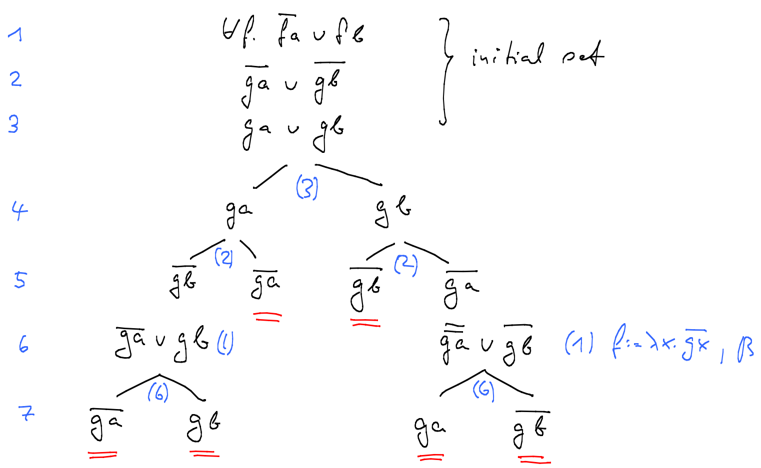
Refutation of $\forall x.\ \overline{fxa} \wedge fbx$

1	$\forall x.\ \overline{fxa} \wedge fbx$	
2	$\overline{fba} \wedge fbb$	(1)
3	$\overline{faa} \wedge fba$	(1)
4	$\overline{fba}$	(2)
5	fba	(3)

Dual: $\exists x.\ fxa \to fbx$

$QL' \vdash s = 0 \Leftrightarrow QL' \vdash \bar{s} = 1$

Refutation of a higher-order set



Trivial Sets

S is trivial if it contains one of the following conflicts:

1) $0 \in S$
2) $\overline{1} \in S$
3) $\exists s:\ s \in S \wedge \overline{s} \in S$
4) $\exists t:\ \overline{t \doteq t} \in S$

$$\boxed{S \text{ trivial} \Rightarrow S \text{ unsatisfiable}}$$

Consistency Classes (CCs)

- A CC is a set of non-trivial sets of normal fofs satisfying certain conditions
- Idea: S has a refutation $\Leftrightarrow$ no CC contains S
- CCs are easier to formalize than refutations
- MET: Every member of a CC is QL'-satisfiable

Definition of Consistency Classes

A set $\mathcal{S}$ of non-trivial sets of normal first-order formulas is a consistency class if

1) $s \wedge t \in S \in \mathcal{S} \Rightarrow S \cup \{s,t\} \in \mathcal{S}$
2) $s \vee t \in S \in \mathcal{S} \Rightarrow S \cup \{s\} \in \mathcal{S} \vee S \cup \{t\} \in \mathcal{S}$

3) $\forall x.s \in S \in \mathcal{S} \wedge s[x:=t]$ first-order $\Rightarrow S \cup \{s[x:=t]\} \in \mathcal{S}$
4) $\exists x.s \in S \in \mathcal{S} \wedge s[x:=a]$ first-order $\wedge\ a \notin \mathcal{N}S \cup \mathcal{B} \Rightarrow S \cup \{s[x:=a]\} \in \mathcal{S}$

5) $S \in \mathcal{S} \wedge s \doteq s$ first-order $\Rightarrow S \cup \{s \doteq s\} \in \mathcal{S}$
6) $\{s_1 \doteq s_2,\ t[x:=s_1]\} \subseteq S \in \mathcal{S} \Rightarrow S \cup \{t[x:=s_2]\} \in \mathcal{S}$

Smullyan's Model Existence Theorem [1963]

> Every frugal member of a CC is satisfiable

S frugal if for every type there are
infinitely many constants not occurring in S

First-Order Compactness

> Let S be set of first-order formulas. Then:
> Every finite subset of S satisfiable $\Rightarrow$ S satisfiable

Proof.

1) $\{S \mid S$ normal and first-order and every finite subset of S is satisfiable$\}$ is a consistency class
2) If S frugal, claim follows with Smullyan's MET
3) If S not frugal, we apply bijective constant renaming Θ such that ΘS frugal and exploit stability. □

First-Order Completeness

> $\{S \mid S$ normal first-order $\wedge$ S $\mathcal{QL}'$-consistent$\}$
> is a consistency class

Proof requires arguments like the following

- $S, s \vee t \vdash^{\mathcal{QL}'} \circ \iff S, s \vdash^{\mathcal{QL}'} \circ \wedge S, t \vdash^{\mathcal{QL}'} \circ$

  Deductivity + tautology $(x \to z) \wedge (y \to z) = x \vee y \to z$

- $S, \exists x.s \vdash^{\mathcal{QL}'} \circ \iff S, s[x:=a] \vdash^{\mathcal{QL}'} \circ$

  if $a \notin \mathcal{V}(S,s) \cup \mathcal{B}$

  see Skolemization

Saturated Sets (S set of normal fofs)

S saturated if the following conditions are satisfied:

1) $s \wedge t \in S \Rightarrow s \in S \wedge t \in S$
2) $s \vee t \in S \Rightarrow s \in S \vee t \in S$
3) $\exists x.s \in S \Rightarrow \exists t : s[x:=t] \in S$
4) $\forall x.s \in S \wedge s[x:=t]$ first-order $\Rightarrow s[x:=t] \in S$
5) $s \doteq s$ first-order $\Rightarrow s \doteq s \in S$
6) $s_1 \doteq s_2 \in S \wedge t[x:=s_1] \in S \Rightarrow t[x:=s_2] \in S$

Hintikkas Model Existence Theorem [1955]

$\boxed{S \text{ saturated and non-trivial} \Rightarrow S \text{ satisfiable}}$

term models are due to Herbrand and Birkhoff

Proof. Construct term model as follows

$[s] = \{t \mid s \doteq t \in S\}$ equivalence class

$\mathfrak{A}C = \{[s] \mid s \doteq s \in S \land s : C\}$ if $C \neq B$

$\mathfrak{A}c\,[s_1]\dots[s_n]$ where $c : C_1 \to \dots \to C_n \to C$

$= [c s_1 \dots s_n]$ if $C \neq B$

$= 1$ if $C = B$ and $c s_1 \dots s_n \in S$

$= 0$ if $C = B$ and $c s_1 \dots s_n \notin S$

Extension Lemma

$\boxed{\text{Every frugal member } S \text{ of a consistency class can be extended to a saturated and non-trivial set of normal first-order terms}}$

Smullyan's MET follows from Extension Lemma and Hintikka's MET

Proof of Extension Lemma

A CC $\mathcal{S}$ is compact if $\forall S:\; S \in \mathcal{S} \Leftrightarrow \forall S' \subseteq_{fin} S : S' \in \mathcal{S}$

Compact CCs are closed under chain limits

$$\boxed{\left.\begin{array}{l} \mathcal{S} \text{ compact} \\ S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots \\ \forall i : S_i \in \mathcal{S} \end{array}\right\} \bigcup_{i \in \mathbb{N}} S_i \in \mathcal{S}}$$

$\boxed{\text{For every CC } \mathcal{S} \text{ exists compact CC } \mathcal{S}'' \text{ such that } \mathcal{S} \subseteq \mathcal{S}''}$

$\mathcal{S}' := \{S \mid \exists S' \in \mathcal{S} : S \subseteq S'\}$ CC, $\mathcal{S} \subseteq \mathcal{S}'$

$\mathcal{S}'' := \{S \mid \forall S' \subseteq_{fin} S : S' \in \mathcal{S}'\}$ CC, $\mathcal{S}' \subseteq \mathcal{S}''$

$\boxed{\mathcal{S} \text{ compact} \land S \in \mathcal{S} \text{ frugal} \Rightarrow \exists S' \in \mathcal{S} : S \subseteq S' \land S' \text{ saturated}}$

Construct chain $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq \bigcup_{i \in \mathbb{N}} S_i = S'$ as follows:

1) Choose enumeration $s_1, s_2, s_3, \dots$ of all fofs

2) $$S_{n+1} = \begin{cases} S_n & \text{if } S_n \cup \{s_{n+1}\} \notin \mathcal{S} \\ S_n \cup \{s_{n+1}\} & \text{otherwise if } s_{n+1} \neq \exists .. \\ S_n \cup \{\exists x s,\, s[x := a]\} & \text{otherwise if } s_{n+1} = \exists x . s \text{ and } a \notin \mathcal{N}(S_n, s) \cup B \end{cases}$$