Higher-Order

Predicate Logic

- PL + axiomatization of quantifiers
- Focus: Deduction techniques for quantifiers

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Specification QL

Extends	PL	
Constants	$\forall_T, \exists_T : (T \to B) \to B$	
Axioms	$\forall(\lambda x.1) = 1$	$\forall 1$
	$\forall f \to fx = 1$	$\forall I$
	$\exists(\lambda x.0) = 0$	$\exists 0$
	$fx \to \exists f = 1$	$\exists I$

polymorphic (Predicate Logic: T is a fixed sort)

Instantiation

axioms are schematic, x, f are variables

Notation: $\forall x.s \rightsquigarrow \forall(\lambda x.s)$

$\exists x.s \rightsquigarrow \exists(\lambda x.s)$

Duality

$\delta 0 = 1$	$\delta 1 = 0$
$\delta \wedge = \vee$	$\delta \vee = \wedge$
$\delta \to = \not\leftarrow$	$\delta \not\leftarrow = \to$
$\delta \leftrightarrow = \not\leftrightarrow$	$\delta \not\leftrightarrow = \leftrightarrow$
$\delta \forall_T = \exists_T$	$\delta \exists_T = \forall_T$

$\delta(\delta s) = s$

$QL \vdash \delta(QL)$

$QL \vdash e \Leftrightarrow QL \vdash \delta e$

Alternative: Axiomatize $\exists$ as defined constant: $\exists f = \overline{\forall x. \overline{fx}}$

Golden Rule

$s \to t = 1 \quad \vdash_{BA} \quad s = s \wedge t \quad \vdash_{BA} \quad t = s \vee t$ GR

Proof. Follows from:

1) $BA \vdash x \to y = x \leftrightarrow x \wedge y$

2) $BA \vdash x \to y = y \leftrightarrow x \vee y$

3) $s = t \vdash_{BA} s \leftrightarrow t = 1$ □

Examples

$QL \vdash \forall f = \forall f \wedge fx$ by $\forall I$, GR

$QL \vdash \exists f = \exists f \vee fx$ by $\exists I$, GR

Drop Laws

can eliminate / introduce quantifiers

$s[x:=t] = 1 \;\vdash_{QL}\; \exists x.s = 1$	Gen ∃	
$s = 1 \;\vdash_{QL}\; \forall x.s = 1$	Gen ∀	Generalisation
$QL \vdash q = \exists x.q$	∃E	Elimination
$QL \vdash q = \forall x.q$	∀E	

Proof of Gen ∀

Claim $\forall x.s = 1 \;\vdash_{QL}\; s = 1$

Proof "⊣" $\forall x.s = \forall x.1$ — $s=1$

$= 1$ — ∀1

"⊢" $s = 1 \wedge s$ — Taut

$= (\forall x.s) \wedge s$ — $\forall x.s = s$

$= (\forall x.s) \wedge (\lambda x.s)x$ — β

$= \forall x.s$ — ∀I, GR

$= 1$ — $\forall x.s = s$ □

Pull Laws (∧,∨)

The following equations are deducible in QL

∀∧	$\forall f \wedge q = \forall x.\, fx \wedge q$	∃∨	$\exists f \vee q = \exists x.\, fx \vee q$
∀∨	$\forall f \vee q = \forall x\, fx \vee q$	∃∧	$\exists f \wedge q = \exists x.\, fx \wedge q$
∀∧'	$\forall f \wedge \forall g = \forall x.\, fx \wedge gx$	∃∨'	$\exists f \vee \exists g = \exists x.\, fx \vee gx$

Proof of ∀∧

Claim $QL \vdash \forall x.\, fx \wedge q = \forall f \wedge q$

Proof $(\forall x.\, fx \wedge q) \Leftrightarrow \forall f \wedge q$ — Eq

Case split on q — BCA

	$q = 0$	$q = 1$	
	$(\forall x.\, fx \wedge 0) \Leftrightarrow \forall f \wedge 0$	$(\forall x.\, fx \wedge 1) \Leftrightarrow \forall f \wedge 1$	
Taut	$(\forall x.0) \Leftrightarrow 0$	$\forall x.\, fx \Leftrightarrow \forall f$	Taut
∀I, GR	$(\forall x.0) \wedge (\lambda x.0)x \Leftrightarrow 0$	$\forall f \Leftrightarrow \forall f$	η
β	$(\forall x.0) \wedge 0 \Leftrightarrow 0$	1	Taut
Taut	1		

□

De Morgan Laws (dM)

$$\overline{\forall x.\, fx} = \exists x.\, \overline{fx}$$

$$\overline{\exists x.\, fx} = \forall x.\, \overline{fx}$$

Proof with MoC and Duality

Proof of dM

MoC $\quad \boxed{\bar{n} = t \;\vdash^{BA}\; n \wedge t = 0,\; n \vee t = 1}$

$$\overline{\exists x.\, fx} = \forall x.\, \overline{fx}$$

MoC

	$(\exists x.\, fx) \wedge \forall x.\, \overline{fx}$	$(\exists x.\, fx) \vee \forall x.\, \overline{fx}$	
$\exists\wedge$	$\exists x.\, fx \wedge \forall x.\, \overline{fx}$	$(\exists x.\, fx) \vee \overline{fx}$	$\forall\vee$, Gen $\forall$
$\forall I$, GR	$\exists x.\, fx \wedge (\forall x.\, \overline{fx}) \wedge \overline{fx}$	$\exists f \vee \overline{fx}$	η
Taut	$\exists x.\, 0$	$fx \to \exists f$	Taut
$\exists 0$	0	1	$\exists I$

Pull Laws ($\to$)

The following equations are deducible in QL

$$q \to \forall f = \forall x.\, q \to fx$$

$$q \to \exists f = \exists x.\, q \to fx$$

$$\forall f \to q = \exists x.\, fx \to q$$

$$\exists f \to q = \forall x.\, fx \to q$$

$$\forall f \to \exists g = \exists x.\, fx \to gx$$

Proof. Use pull laws for $\wedge, \vee$ and dM.

Turing's Law

$$QL \vdash \overline{\exists x \forall y.\, fxy \Leftrightarrow \overline{fyy}} = 1$$

$f : T \to T \to B$

1) There is no barber who shaves everyone who doesn't shave himself

2) There is no TM that halts on the rep of a TM y if and only if y doesn't halt on its own rep

3) There is no set that contains all sets that don't contain themselves

Turing's Law

$$QL \vdash \overline{\exists x \forall y.\ fxy \Leftrightarrow \overline{fyy}} = 1$$

$f : T \to T \to B$

backward proof!

Proof

$\overline{\exists x \forall y.\ fxy \Leftrightarrow \overline{fyy}}$	
$\forall x \exists y.\ \overline{fxy \Leftrightarrow \overline{fyy}}$	dM
$\exists y.\ fxy \Leftrightarrow fyy$	Gen $\forall$, Taut
$fxx \Leftrightarrow fxx$	Gen $\exists$
1	Taut

Cantor's Law

Let $\mathcal{X}$ be a set. Then there exists no surjective function $\mathcal{X} \to \mathcal{P}\mathcal{X}$

$\mathcal{X} \leadsto$ type T

$\mathcal{P}\mathcal{X} \leadsto$ type $T \to B$

$$\overline{\exists f \forall g \exists x.\ fx \doteq g}$$

$f : T \to (T \to B)$, $g : T \to B$, $x : T$

Cantor's Law (for $\mathcal{X} = B$)

$$PL \vdash \overline{\exists f \forall g \exists x \forall y.\ fxy \Leftrightarrow gy} = 1$$

$f : B \to B \to B$

$g : B \to B$

$x, y : B$

Proof

$\overline{\exists f \forall g \exists x \forall y.\ fxy \Leftrightarrow gy}$	
$\forall f \exists g \forall x \exists y.\ fxy \Leftrightarrow \overline{gy}$	dM, Taut
$\forall x \exists y.\ fxy \Leftrightarrow (\overline{\lambda y.\ \overline{fyy})\, y}$	Gen $\forall$, Gen $\exists$ $(g := \lambda y.\ \overline{fyy})$
$\exists y.\ fxy \Leftrightarrow fyy$	Gen $\forall$, β, Taut
$fxx \Leftrightarrow fxx$	Gen $\exists$
1	Taut