Assignment 7
Introduction to Computational Logic, SS 2008

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Read in the lecture notes: Chapter 6

Exercise 7.1 (Termination of $\mathcal{T}_{\mathrm{PLN}}$ on $\mathrm{PLN}_c^{\exists *}$) For finite subsets A and A' of $\mathrm{PLN}_c^{\exists *}$ let $A \xrightarrow{\mathcal{T}_{\mathrm{PLN}}} A'$ be the relation that holds if there is a refutation step which is an instance of a rule from $\mathcal{T}_{\mathrm{PLN}}$ where $A \vdash \bot$ is the conclusion of the step and $A' \vdash \bot$ is one of the premisses. Stated in terms of tableaux, $A \xrightarrow{\mathcal{T}_{\mathrm{PLN}}} A'$ if we can extend a branch with formulas A to a branch with formulas A' (possibly creating other branches as well) using a rule from $\mathcal{T}_{\mathrm{PLN}}$. Recall the definition of $\mathrm{Stock}(A)$ and $\mathrm{Slack}(A)$ from the lecture and the lecture notes. For any A let $\exists(A)$ be

$$\{\exists x.s \in \mathrm{Sub}(A) \mid \forall a \in \mathcal{P}.s^x_a \notin A\}.$$

a) Define a natural number $\mathrm{Power}^{\exists}(A)$ for finite sets $A \subseteq \mathrm{PLN}_c^{\exists *}$ making use of the set $\exists(A)$.

b) Prove that if $A \xrightarrow{\mathcal{T}_{\mathrm{PLN}}} A'$ via the **Exists** rule, then

$$\mathrm{Power}^{\exists}(A) > \mathrm{Power}^{\exists}(A')$$

c) Prove that if $A \xrightarrow{\mathcal{T}_{\mathrm{PLN}}} A'$ via the **And** or **Or** rule, then

$$\mathrm{Power}^{\exists}(A) \geq \mathrm{Power}^{\exists}(A')$$

d) Prove that if $A \xrightarrow{\mathcal{T}_{\mathrm{PLN}}} A'$ via the **Forall** rule, then

$$\mathrm{Power}^{\exists}(A) \geq \mathrm{Power}^{\exists}(A')$$

e) Conclude that $\xrightarrow{\mathcal{T}_{\mathrm{PLN}}}$ terminates on $\mathrm{PLN}_c^{\exists *}$.

Exercise 7.2 (Subset Closure) Let C be a consistency class. Define

$$C^+ = \{A \mid \exists A' \in C.A \subseteq A'\}.$$

Prove C^+ is a subset closed consistency class.

Exercise 7.3 Recall that C_{PLN} is defined to be the set of all finite $A \subseteq \mathrm{PLN}_c$ such that there is no refutation $A \vdash \bot$ using $\mathcal{T}_{\mathrm{PLN}}$-rules. Prove C_{PLN} is a consistency class.

Exercise 7.4 Let $p : IIB$ be a name, $a, b \in \mathcal{P}$ and $z, w \in \mathcal{V}$. Use Jitpro to give complete tableau refutations for the following set A of **PLN**$_c$-formulas:

$$\begin{array}{lll} A & = \{ & \forall z.(\neg pzb \vee \forall w.\neg pwz \vee pwa) \wedge ((\exists w.pwz \wedge \neg pwa) \vee pzb), \\ & & \neg pab \\ & & \} \end{array}$$

(See Jitpro Exercise 3.1 in Jitpro Exercises 3 on the services page.) Then draw the complete tableau on paper using only the rules from $\mathcal{T}_{\mathbf{PLN}}$.

Exercise 7.5 Let $p : IIB$ be a name, $a, b, c \in \mathcal{P}$ and $x, y, z, w \in \mathcal{V}$. Use Jitpro to give complete tableau refutations for the following sets of **PLN**$_c$-formulas. (See Jitpro Exercises 3.2, 3.3 and 3.4 on the services page.) You do not need to draw the tableau proofs on paper. **Warning:** These may be challenging. If you have trouble, post questions to the discussion board. We are willing to give hints!

- $$\begin{array}{lll} A & = \{ & \forall z.(\neg pzb \vee \forall w.\neg pwz \vee pwa) \wedge ((\exists w.pwz \wedge \neg pwa) \vee pzb), \\ & & \forall z.(\neg pzc \vee \exists w.pwb \wedge pzw) \wedge ((\forall w.\neg pwb \vee \neg pzw) \vee pzc), \\ & & \exists w.pwc \wedge \neg pwa \\ & & \} \end{array}$$

- $$\begin{array}{lll} A & = \{ & \forall z.(\neg pzb \vee \forall w.\neg pwz \vee pwa) \wedge ((\exists w.pwz \wedge \neg pwa) \vee pzb), \\ & & \forall z.(\neg pzc \vee \exists w.pwb \wedge pzw) \wedge ((\forall w.\neg pwb \vee \neg pzw) \vee pzc), \\ & & \exists w.pwa \wedge \neg pwc \\ & & \} \end{array}$$

- $$\begin{array}{lll} A & = \{ & \exists y.\forall x.\neg pxy, \\ & & \forall x.\exists y.\forall z.(\neg pzy \vee \forall w.\neg pwz \vee pwx) \wedge ((\exists w.pwz \wedge \neg pwx) \vee pzy), \\ & & \forall y.\exists x.pxy \wedge (\exists z.pzx) \vee (\forall z.\neg pzx) \wedge \neg pxy \\ & & \} \end{array}$$