Midterm Exam
Introduction to Computational Logic

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Name	Seat
Matriculation	Code

Please put your ID (or passport) and your student card on your desk.

Open the exam booklet only after you have been asked to do so. After you have opened the exam booklet, it is your obligation to check whether it is complete.

You may only use the exam booklet that carries your name and matriculation. You have to write the exam on the seat with the number that is printed on your exam booklet.

No auxiliary means are allowed. At your desk, you may only have writing utensils, beverages, food, and ID cards. Bags and jackets have to be left at the walls of the lecture room.

If you leave the room without turning in your exam booklet, then this will be judged as an attempt of deception.

If you need to go to the bathroom during the exam, please turn in your exam booklet. Only one person may go to the bathroom at a time.

All solutions have to be written on the right hand side pages of the exam booklet. The empty left hand side pages may serve as draft paper and **will not be graded**. No other paper is admitted. You may use a pencil.

The exam lasts 120 minutes. You can obtain at most 120 points. The number of points you can get for a problem gives you a hint about how much time you should spend on that problem. For passing the exam it is sufficient to obtain 60 points.

Every attempt of deception will force us to exclude you from this exam and all following exams of this course. The university keeps a record of attempts of deception.

1	2	3	4	5	6	7	8	9	10	Sum	**Grade**
11	12	9	16	14	10	12	12	12	12	120	

Problem 1. Locally Nameless Representation, 11 points

Draw the LNR of the formula $\forall x\,\forall y\,\exists z.\ x + z = y$. Assume $+ : NNN$ and annotate each λ-node with its argument type.

Problem 2. Choice, 12 points

You are given addition $(+)$ and a choice function $C \in (\mathbb{Z} \to \mathbb{B}) \to \mathbb{Z}$ for the integers. Express the following with the logical operations and the given primitives.

a) Zero $0 \in \mathbb{Z}$.

b) Subtraction $- \in \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$.

c) Conditional $\mathsf{cond} \in \mathbb{B} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$.

Problem 3. Relations and Quantifiers, $9 = 3 \cdot 3$ points

For each of the following formulas, draw a finite relation that satisfies it.

a) $\forall x\ \exists y.\ x \neq y \wedge rxy$

b) $\exists xyz.\ rxy \wedge ryz \wedge \neg rxz$

c) $\forall x.\ rxx \wedge \exists y.\ x \neq y \wedge rxy$

Problem 4. Quickies, $16 = 4 \cdot 4$ points

a) Give a closed term that has the type $(\alpha\beta\gamma)(\alpha\beta)\alpha\gamma$.

b) Give an untyped term s such that $\{\, t \mid s \to_\beta^* t \,\}$ is infinite.

c) Give all contexts C such that there exists a β- or η-redex s such that $C[s] = \lambda x.(\lambda x.x)x$.

d) Draw a finite relation that is not confluent and not terminating.

Problem 5. Natural Numbers, 14 points

Specify the natural numbers with $O : N$, $S : NN$, and $\leq : NNB$.

Problem 6. Reachability, 10 points

Give a term $t : (II)IB$ such that $\mathcal{I} \vDash tfx$ if and only if

$$\mathcal{I}I = \{\mathcal{I}x,\ \mathcal{I}(fx),\ \mathcal{I}(f(fx)),\ \ldots\}$$

for every interpretation $\mathcal{I}$ and some names $f : II$ and $x : I$.

Problem 7. Basic Proof System, 12 points

Derive $\dfrac{\vdash fxy = s}{\vdash fx = (\lambda xy.s)x}$ in **B**.

Problem 8. Natural Deduction, 12 points

Derive $\vdash (\forall x.f x) \rightarrow \exists x.f x$ in **ND.**

Problem 9. Propositional Tableau, 12 points

Give a tableau that proves $\vdash x \rightarrow y \equiv \neg y \rightarrow \neg x$.

Problem 10. First-Order Tableau, 12 points

Give a tableau that proves $\vdash (\forall x. f x) \wedge (\forall x. g x) \rightarrow \forall x.\ f x \wedge g x$.

Basic Proof System B

Triv $\dfrac{}{A,\, s \vdash s}$ **Weak** $\dfrac{A \vdash s}{B \vdash s}\ A \subseteq B$ **Sub** $\dfrac{A \vdash s}{\theta A \vdash \theta s}$ **Lam** $\dfrac{A \vdash s}{A \vdash t}\ \ s \sim_\lambda t$

Ded $\dfrac{A, s \vdash t}{A \vdash s \to t}$ **MP** $\dfrac{A \vdash s \to t \qquad A \vdash s}{A \vdash t}$

Ref $\dfrac{}{A \vdash s{=}s}$ **Rew** $\dfrac{A \vdash s{=}t \qquad A \vdash C[s]}{A \vdash C[t]}$ C admissible for A

D⊤ $\vdash \top \equiv \bot{\to}\bot$ **D¬** $\vdash \neg x \equiv x{\to}\bot$

D∨ $\vdash x{\vee}y \equiv \neg x{\to}y$ **D∧** $\vdash x{\wedge}y \equiv \neg(\neg x{\vee}\neg y)$

D∀ $\vdash \forall f \equiv f{=}\lambda x.\top$ **D∃** $\vdash \exists f \equiv \neg\forall x.\neg f x$

BCA $f\bot,\ f\top \vdash fx$ **Choice** $\vdash \exists c.\forall f.\ \exists f \to f(cf)$

Proof System ND

Triv $\dfrac{}{A,\ s \vdash s}$ **Weak** $\dfrac{A \vdash s}{B \vdash s}\ A \subseteq B$

Ded $\dfrac{A, s \vdash t}{A \vdash s{\to}t}$ **MP** $\dfrac{A \vdash s{\to}t \qquad A \vdash s}{A \vdash t}$

I∧ $\dfrac{A \vdash s_1 \qquad A \vdash s_2}{A \vdash s_1 \wedge s_2}$ **E∧** $\dfrac{A \vdash s_1 \wedge s_2}{A \vdash s_i}$

I∨ $\dfrac{A \vdash s_i}{A \vdash s_1 \vee s_2}$ **E∨** $\dfrac{A \vdash s_1 \vee s_2 \qquad A, s_1 \vdash t \qquad A, s_2 \vdash t}{A \vdash t}$

I∀ $\dfrac{A \vdash s^x_y}{A \vdash \forall x.s}\ y \notin \mathcal{N}A \cup \mathcal{N}(\forall x.s)$ **E∀** $\dfrac{A \vdash \forall x.s}{A \vdash s^x_t}$

I∃ $\dfrac{A \vdash s^x_t}{A \vdash \exists x.s}$ **E∃** $\dfrac{A \vdash \exists x.s \qquad A, s^x_y \vdash t}{A \vdash t}\ y \notin \mathcal{N}A \cup \mathcal{N}(\exists x.s) \cup \mathcal{N}t$

CB $\dfrac{A,\ s{\to}\bot \vdash \bot}{A \vdash s}$

Tableau Rules

$$\textbf{Closed}\ \frac{s, \neg s}{\emptyset} \qquad \textbf{Closed}\ \frac{\bot}{\emptyset} \qquad \textbf{Closed}\ \frac{\neg\top}{\emptyset} \qquad \textbf{Closed}\ \frac{s \neq s}{\emptyset}$$

$$\textbf{DNeg}\ \frac{\neg\neg s}{s} \qquad \textbf{DeMorgan}\ \frac{\neg(s \vee t)}{\neg s \wedge \neg t} \qquad \textbf{DeMorgan}\ \frac{\neg(s \wedge t)}{\neg s \vee \neg t}$$

$$\textbf{DeMorgan}\ \frac{\neg\forall x.s}{\exists x.\neg s} \qquad \textbf{DeMorgan}\ \frac{\neg\exists x.s}{\forall x.\neg s} \qquad \textbf{And}\ \frac{s \wedge t}{s, t}$$

$$\textbf{Or}\ \frac{s \vee t}{s \mid t} \qquad \textbf{Imp}\ \frac{s \rightarrow t}{t \mid \neg s} \qquad \textbf{NegImp}\ \frac{\neg(s \rightarrow t)}{s, \neg t}$$

$$\textbf{Boolean=}\ \frac{s \equiv t}{s, t \mid \neg s, \neg t} \qquad \textbf{Boolean}\neq\ \frac{s \not\equiv t}{s, \neg t \mid t, \neg s}$$

$$\textbf{Forall}\ \frac{\forall x.s}{s^x_t} \qquad \textbf{Exists}\ \frac{\exists x.s}{s^x_y}\ y \text{ fresh}$$