Lemma 2.2.1 (Generalized Cut) If $A \Rightarrow s$ and $B \Rightarrow u$, then $A + (B \setminus s) \Rightarrow u$.

Proof If we try an induction on $A \Rightarrow s$, all cases go through but the case for the right implication rule. The problem can be fixed by adding an outermost induction on s, which gives us strong inductive hypotheses for the case $s = (s_1 \to s_2)$.

For each case of the inner inductions on $A \Rightarrow s$ we have the assumption $B \Rightarrow u$ and the claim $A + (B \setminus s) \Rightarrow u$.

Let $A \Rightarrow s$ be obtained with the variable rule. Then $s = x \in A$. Hence $B \subseteq A + (B \setminus s)$ and the claim follows by weakening from $B \Rightarrow u$.

Let $A \Rightarrow s$ be obtained with the falsity rule. Then $\bot \in A + (B \setminus s)$ and the claim follows with the falsity rule.

Let $A \Rightarrow s$ be obtained with the left implication rule. Then $t_1 \to t_2 \in A$, $A \Rightarrow t_1$, and $A, t_2 \Rightarrow s$. The inductive hypothesis for $A, t_2 \Rightarrow s$ is $A, t2 + (B \setminus s) \Rightarrow u$. We obtain the claim with the left implication rule for $t_1 \to t_2$, which leaves us with the proof obligations $A + (B \setminus s) \Rightarrow t_1$ and $A + (B \setminus s), t_2 \Rightarrow u$. The obligations are easily obtained with weakening.

Let $A \Rightarrow s$ be obtained with the right implication rule. We have $s = (s_1 \to s_2)$ and $A, s_1 \Rightarrow s_2$. The inductive hypothesis for $A, s_1 \Rightarrow s_2$ will not be used. The inductive hypotheses for s_1 and s_2 are as follows:

- For all A, B and u, if $A \Rightarrow s_1$ and $B \Rightarrow u$, then $A + B \setminus s_1 \Rightarrow u$.
- For all A, B and u, if $A \Rightarrow s_2$ and $B \Rightarrow u$, then $A + B \setminus s_2 \Rightarrow u$.

We prove the claim $A + B \setminus s \Rightarrow u$ by induction on $B \Rightarrow u$. This is the final induction.

We consider the case for the left implication rule, the other cases are straightforward. We have $t_1 \to t_2 \in B$, $B \Rightarrow t_1$, and $B, t_2 \Rightarrow u$. The inductive hypotheses for $B \Rightarrow t_1$ and $B, t_2 \Rightarrow u$ are as follows:

- IH1: $A + B \setminus s \Rightarrow t_1$
- IH2: $A + (B, t_2) \setminus s \Rightarrow u$

We prove the claim $A + B \setminus s \Rightarrow u$ by case analysis.

1. Let $s_1 = t_1$ and $s_2 = t_2$. The application of the inductive hypothesis for s_1 to IH1 and $A, s_1 \Rightarrow s_2$ yields
$$A + B \setminus s + (A, s_1) \setminus s_1 \Rightarrow s_2$$
The application of the inductive hypothesis for s_2 to the above statement and IH2 yields
$$A + B \setminus s + (A, s_1) \setminus s_1 + (A + (B, s_2) \setminus s) \setminus s_2 \Rightarrow u$$
The claim now follows with weakening.
2. Let $s \neq (t_1 \to t_2)$. Then $(t_1 \to t_2) \in A + B \setminus s$. We obtain the claim with the left implication rule. This leaves us with the proof obligations $A + B \setminus s \Rightarrow t_1$ and $A + B \setminus s, t_2 \Rightarrow u$, which follow with IH1, IH2, and weakening. ∎