Set Theory

Lecture Notes

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1 Introduction

A set theory is an axiomatic theory that establishes a type of sets. The goal is to have enough sets such that every mathematical object can be represented as a set. Axiomatic set theories are closed in that the elements of sets must be sets.

We consider axiomatic set theories that appear as subtheories of the standard set theory ZFC [11, 5, 12] (Zermelo Frankel with choice). We formalize our set theories in constructive type theory and make explicit which results require excluded middle. We use inductive definitions to define the classes of hereditarily finite sets, well-founded sets, and ordinals. This is a substantial deviation from the standard presentation of ZFC based on classical first-order logic.

The textbook of Hrbacek and Jech [6] is a mathematical introduction to set theory leaving the logical basis implicit. Devlin's [4] textbook says more about the logical basis (first-order logic) and covers non-well-founded sets. Barras [2] formalizes axiomatic set theory in Coq.

2 Sets and Classes

The basic setup of our set theories is straightforward: We assume a type *set* and a binary predicate $\in$ on *set* modelling set membership.

$$\mathit{set} : \mathit{Type}$$
$$\in \;:\; \mathit{set} \to \mathit{set} \to \mathit{Prop}$$

This setup is distinguished from a naive view of sets in that the elements of sets must be sets. The letters x, y, z, and u will range over sets in the following.

A **class** (of sets) is a unary predicate on sets. The letters p and q will range over classes. A set x **realizes** a class p if the elements of x are exactly the sets satisfying p:

$$\textit{realizes}\; x\; p \;:=\; \forall z.\, z \in x \leftrightarrow pz$$

There is a straightforward unrealizable class discovered by Bertrand Russell [10].

Fact 1 (Russell) The class $\lambda x.x \notin x$ is not realizable.

Proof Assume y realizes $\lambda x.x \notin x$. Then $y \in y \leftrightarrow y \notin y$. Contradiction. ∎

Note that the proof does not use extensionality. We can reformulate Russel's fact such that it doesn't use the notion of a class.

Fact 2 (Russell)
There is no set containing exactly those sets that do not contain themselves.

Russell's fact is known as *Russel's paradox*. If formulated in words without proper logical foundation, Russel's paradox can be quite confusing. This was the situation when Russel discovered the fact in 1901. Russel's paradox made it clear that abstract mathematics needs a logical foundation. Since such a logical foundation did not exist at the time, mathematicians had to admit a foundational crisis (Grundlagenkrise). Incidentally, Russel's paradox was discovered by Ernst Zermelo before Russel in 1900. Zermelo did not publish his insight.

When Cantor [3] started set theory in the late 19th century, he was naive about the existence of sets. He more or less assumed that every collection of objects can be assembled into a set. Cantor [3] writes: *Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen.*

3 Extensionality and Set Inclusion

A basic property of sets is **extensionality**: Two sets are equal if they have the same elements. We assume extensionality.

$$\textit{Ext} \;:\; \forall xy.\, (\forall z.\, z \in x \leftrightarrow z \in y) \rightarrow x = y$$

Extensionality means that a set is fully determined by its elements.

We define **set inclusion** as follows:

$$x \subseteq y \;:=\; \forall z.\, z \in x \rightarrow x \in y$$

Fact 3 $x = y$ if and only if $x \subseteq y$ and $y \subseteq x$.

Proof Follows with extensionality. ∎

Fact 4 Set inclusion is a partial order on sets.

We say that a set x **is in a class** p if px holds. We write $x \subseteq p$ and say that x is a **subset** of p if every element of x is in p.

Fact 5 (Criteria for Unrealizability) Let p be a class such that for every subset $x \subseteq p$ there exists a set y in p such that $y \notin x$. Then p is unrealizable.

A set in a class is called the **least element** of the class if it is a subset of every element of the class. A set in a class is called the **greatest element** of the class if it is a superset of every element of the class. The use of the definite article for least and greatest elements is justified by their uniqueness.

Fact 6 (Uniqueness of Least and Greatest Elements)
A class has at most one least and at most one greatest element.

4 Empty Set

We need axioms to establish the existence of sets. With our assumptions so far, we cannot prove that a set exists. Given Russel's paradox, we need to be careful when we assume the existence of sets. Axiomatic set theory asserts the existence of certain sets with carefully chosen axioms. Axiomatic set theory was started by Ernst Zermelo [11] in 1908. In the 1920's, a now standard axiomatization of set theory, ZF for Zermelo and Fraenkel, was established.

We start with two axioms saying that an empty set exists.

$$\emptyset \;:\; \mathit{set}$$
$$\mathit{Eset} \;:\; \forall x.\, x \notin \emptyset$$

Fact 7 (Uniqueness) $\emptyset$ is the only set containing no elements.

Proof Follows with extensionality. ∎

A set is **inhabited** if it has an element.

$$\mathit{inhabited}\ x \;:=\; \exists z.\, z \in x$$

Fact 8 A set x is not inhabited if and only if $x = \emptyset$.

Fact 9 If $x \subseteq \emptyset$, then $x = \emptyset$.

5 Unordered Pairs, Singletons, Ordered Pairs

We assume a function that, given two sets x and y, yields the set containing exactly x and y.

$$\begin{aligned} \mathit{upair} &: \mathit{set} \to \mathit{set} \to \mathit{set} \\ \mathit{Upair} &: \forall xyz.\ z \in \mathit{upair}\ x\ y \leftrightarrow z = x \vee z = y \end{aligned}$$

Sets of the form $\mathit{upair}\ x\ y$ are called **unordered pairs**. We define **singletons** as follows:

$$\{x\} := \mathit{upair}\ x\ x$$

Fact 10 (Symmetry) $\mathit{upair}\ x\ y = \mathit{upair}\ y\ x$.

We will use the notation

$$\{x, y\} := \mathit{upair}\ x\ y$$

The expectations associated with this familiar notation (e.g., $\{x, y\} = \{y, x\}$ and $\{x, x\} = \{x\}$) are justified by the above fact and the notation for singletons.

Fact 11 $x \in \{x, y\}$ and $y \in \{x, y\}$.

Fact 12 $\{x, y\} \neq \emptyset$.

Fact 13 (Injectivity)
If $\{x, y\} = \{a, b\}$, then either $x = a$ and $y = b$ or $x = b$ and $y = a$.

Fact 14 If $\{x\} = \{a, b\}$, then $x = a = b$.

Fact 15 (Injectivity) If $\{x\} = \{y\}$, then $x = y$.

Fact 16 If $x \in \{y\}$, then $x = y$.

Fact 17 If $x \subseteq \{y\}$ and x is inhabited, then $x = \{y\}$.

Although sets are unordered, they can represent ordered pairs. Following Kuratowski [7], we define **ordered pairs** as follows:

$$(x, y) := \{\{x\}, \{x, y\}\}$$

Fact 18 (Injectivity) If $(x, y) = (a, b)$, then $x = a$ and $y = b$.

Proof Follows with the injectivity facts stated above. ∎

6 Union

We assume a function that, given a set x, yields the union of all sets that are elements of x (recall that the elements of sets are sets).

$$\begin{aligned} \bigcup &: \mathit{set} \to \mathit{set} \\ \mathit{Union} &: \forall xz.\ z \in \bigcup x \leftrightarrow \exists y \in x.\ z \in y \end{aligned}$$

Fact 19 $\bigcup x \subseteq u$ if and only if every element of x is a subset of u.

Fact 20 $y \subseteq \bigcup x$ if y is an element of x.

Fact 21 (Least Upper Bound Characterization)
$\bigcup x = u$ if and only if the following conditions hold:

1. Every element of x is a subset of u.
2. u is a subset of every set that is a superset of every element of x.

Fact 22 $\bigcup \emptyset = \emptyset$, $\bigcup\{x\} = x$, and $\bigcup(x, y) = \{x, y\}$.

Fact 23 (Monotonicity) $\bigcup x \subseteq \bigcup y$ if $x \subseteq y$.

We define **binary union** as follows:

$$x \cup y \ := \ \bigcup\{x, y\}$$

Fact 24 $z \in x \cup y$ if and only if $z \in x$ or $z \in y$.

Fact 25

1. $u \subseteq x \cup y$ if $u \subseteq x$ or $u \subseteq y$.
2. $x \cup y \subseteq u$ if $x \subseteq u$ and $y \subseteq u$.

7 Adjunction and Finite Sets

We define **adjunction** as follows:

$$x\,;y \ := \ x \cup \{y\}$$

Fact 26 $z \in x\,;y$ if and only if $z \in x$ or $z = y$.

Fact 27 $x \subseteq x\,;y$ and $y \in x\,;y$.

Fact 28

1. $\emptyset \cup x = x$

2. $\{x, y\} = (\emptyset\,;x)\,;y$
3. $(x\,;y) \cup z = (x \cup z)\,;y$
4. $\bigcup x \cup y = \bigcup(x\,;y)$

We define a function *fin* mapping lists of sets to sets:

$$\begin{aligned} \mathit{fin}\ \mathit{nil} &:= \emptyset \\ \mathit{fin}\ (x :: A) &:= \mathit{fin}\ A\,;x \end{aligned}$$

We call a set x **finite** if $x = \mathit{fin}\ A$ for some list of sets A. Formally, we define the **class of finite sets** as follows:

$$\mathit{Fin}\ x := \exists A.\ x = \mathit{fin}\ A$$

Fact 29

1. $x \in \mathit{fin}\ A$ if and only if $x \in A$.
2. $\mathit{fin}\ A \subseteq \mathit{fin}\ B$ if and only if $A \subseteq B$.
3. $\mathit{fin}\ A = \mathit{fin}\ B$ if and only if $A \equiv B$.
4. $\mathit{fin}\ (A + B) = \mathit{fin}\ A \cup \mathit{fin}\ B$.

We can now define the common notation for finite sets:

$$\{x_1, \ldots, x_n\} := \mathit{fin}\ [x_1, \ldots, x_n]$$

The new definition of the curly braces notation subsumes the previous definitions for unordered pairs and singletons. What we have arrived at is an explanation of finite sets and the curly braces notation in terms of lists, the empty set, and adjunction.

One would expect that every subset of a finite set is finite. Without further assumptions a proof confirming this expectation seems impossible. A proof is possible with excluded middle.

8 Numerals and Transitive Sets

We now come to an ingenious representation of the natural number due to John von Neumann [8]. As one would expect, the empty set serves as representation of 0. As successor function we take self adjunction:

$$\sigma x := x\,;x \qquad\qquad \textbf{successor}$$

For every natural number, we define a set called the **numeral** for the number:

$$\overline{n} := \sigma^n\,\emptyset$$

The first 3 numerals are

$$
\begin{aligned}
&\emptyset \\
&\{\emptyset\} \\
&\{\emptyset, \{\emptyset\}\} \\
&\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}
$$

Fact 30

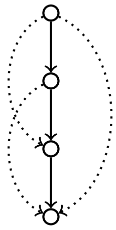
1. $z \in \sigma x$ if and only if $z \in x$ or $z = x$.
2. $x \in \sigma x$ and $x \subseteq \sigma x$.

Fact 31 A numeral $\overline{n}$ is the set of all numerals $\overline{m}$ such that $m < n$. More precisely:

1. If $m < n$, then $\overline{m} \in \overline{n}$.
2. If $x \in \overline{n}$, then $x = \overline{m}$ for some $m < n$.

The membership predicate organizes the type of sets into an infinite graph whose nodes are the sets and whose edges describe the membership relation. We direct the edges such that an edge from x to y says that y is an element of x. Note that the empty set is the single terminal node of the graph. We speak of the graph view of sets.

The **graph of a set** is the graph reachable from the set. Note that the graphs of the numerals are finite. Moreover, the edge relation of the graphs of the numerals is transitive. Here is the graph of the numeral for 3.



The dotted edges can be obtained as the transitive closure of the solid edges. Note that the numerals for 0, 1, and 2 appear as subgraphs.

The transitivity property of numerals turns out to be essential. Following the literature, we define transitivity of sets as follows.

A set x is **transitive** if every element of x is a subset of x (i.e., $y \subseteq x$ for every $y \in x$).

Fact 32 A set x is transitive if and only if every element of every element of x is an element of x.

It turns out that there are finite transitive sets whose graph is not transitive (see Exercise 40).

Fact 33 (Transitivity of Numerals)

1. $\emptyset$ is transitive.
2. σx is transitive if x is transitive.
3. Every numeral is transitive.

Fact 34 (Predecessor) If x is transitive, then $\bigcup \sigma x = x$.

Fact 35 (Injectivity) If x and y are transtive and $\sigma x = \sigma y$, then $x = y$.

Theorem 36 (Adequacy) For all numbers m and n:

1. $m = n$ if and only if $\overline{m} = \overline{n}$.
2. $m < n$ if and only if $\overline{m} \in \overline{n}$.
3. $m \leq n$ if and only if $\overline{m} \subseteq \overline{n}$.

Proof Claim (1) follows by induction on n with the injectivity of σ and the transitivity of numerals. Claim (2) follows with Fact 31 and Claim (1). Claim (3) follows with Fact 31 and Claim (2). ■

Fact 37 $\bigcup x$ is transitive if every element of x is transitive.

Fact 38 $\bigcup x \subseteq x$ if and only if x is transitive.

Exercise 39 Show $\overline{n} \notin \overline{n}$.

Exercise 40 We define the **singleton numerals** as follows:

$$\begin{aligned} \tilde{0} &:= \emptyset \\ \widetilde{Sn} &:= \{\tilde{n}\} \end{aligned}$$

a) Show that the singleton numerals for different numbers are different sets.
b) Draw the graphs of $\tilde{3}$ and $\overline{3}$ and convince yourself that the graph of $\overline{3}$ is the transitive closure of the graph of $\tilde{3}$.
c) Give a transitive set whose graph is not transitive.

9 Hereditarily Finite Sets

We define the class of **hereditarily finite sets** inductively:

1. $\emptyset$ is hereditarily finite.

2. $x\,;y$ is hereditarily finite if x and y are hereditarily finite.

We use *HF* to denote the class of hereditarily finite sets. The inductive definition of *HF* can be represented with two rules:

$$\frac{}{HF\ \emptyset} \qquad\qquad \frac{HF\ x \qquad HF\ y}{HF\ (x\,;y)}$$

Fact 41 Numerals are hereditarily finite.

Fact 42 Every hereditarily finite set is finite.

Note that the graph of a hereditarily finite set is finite. Moreover, the graphs of hereditarily finite sets are loop free.

Fact 43 Every element of a hereditarily finite set is hereditarily finite.

Proof Let *HF* x and $y \in x$. We show *HF* y by induction on *HF* x.

If $x = \emptyset$, the claim follows by contradiction with $y \in x$.

If $x = x'\,;z$, then *HF* x', *HF* z, and either $y \in x'$ or $y = z$. If $y \in x'$, the claim follows by the inductive hypothesis. ■

Fact 44 A set is hereditarily finite if and only if it is finite and all its elements are hereditarily finite.

Fact 45 (Closure Properties)

1. $\{x, y\}$ is hereditarily finite if x and y are hereditarily finite.
2. $x \cup y$ is hereditarily finite if x and y are hereditarily finite.
3. $\bigcup x$ is hereditarily finite if x is hereditarily finite.

Proof The equations from Fact 28 are useful. Claims (2) and (3) follow by induction on *HF*. ■

Exercise 46 We define a class *HFL* inductively with the following rule saying that a finite set is in *HFL* if each of its elements is in *HFL*:

$$\frac{\forall x \in A.\ HFL\ x}{HFL\ (fin\ A)}$$

Prove that the classes *HF* and *HFL* are equivalent.

Exercise 47 Prove that membership, inclusion, and equality of hereditarily finite sets is logically decidable. Proceed as follows.

a) Prove that $x \in \mathit{fin}\ A$ is logically decidable if $x = y$ is logically decidable for every $y \in A$.

b) Prove that $\mathit{fin}\ A \subseteq \mathit{fin}\ B$ is logically decidable if $x = y$ is logically decidable for every $x \in A$ and every $y \in B$.

c) Prove that $\mathit{fin}\ A = \mathit{fin}\ B$ is logically decidable if $x = y$ is logically decidable for every $x \in A$ and every $y \in B$.

d) Prove that membership, inclusion, and equality of HFL sets is logically decidable.

e) Prove that membership, inclusion, and equality of hereditarily finite sets is logically decidable.

10 Well-Founded Sets

We say that a set is *well-founded* if the graph of the set does not contain an infinite path. Put differently, a set is well-founded if every process descending down the membership edges always terminates. Intuitively it is clear that every hereditarily finite set if well-founded.

In traditional mathematical theories only well-founded sets are used. The standard set theory ZF has in fact an axiom saying that every set is well-founded. This axiom is called *regularity axiom* and was introduced by von Neumann [9]. We will not impose the regularity axiom and thus regard a set theory that is more general than ZF. Peter Aczel [1] has developed an interesting set theory equipped with an axiom asserting the existence of certain non-well-founded sets.

Our characterizations of well-founded sets stated so far provide useful intuitions but do not suffice for a precise mathematical definition. Our official definition of well-founded sets is inductive and surprisingly straightforward.

A set is **well-founded** if all its elements are well-founded.

We use *WF* to denote the class of all well-founded sets. We may represent the inductive definition of *WF* with a single rule:

$$\frac{x \subseteq \mathit{WF}}{\mathit{WF}\ x}$$

The inductive definition of *WF* provides an induction principle that will be useful: To show that every well-founded set satisfies a property p, it suffices to show that a set of well-founded sets satisfies p if all its elements satisfy p. We will refer to this induction principle as **well-founded induction**.

Fact 48 A set is well-founded if and only if each of its elements is well-founded.

Fact 49 Every subset of a well-founded set is well-founded.

Fact 50 If $\bigcup x$ is well-founded, then x is well-founded.

Proof Let $\bigcup x$ be well-founded and $y \in x$. It suffices to show that y is well-founded. This follows from $y \subseteq \bigcup x$. ∎

Fact 51 (Closure Properties)

1. $\{x, y\}$ is well-founded if x and y are well-founded.
2. $\bigcup x$ is well-founded if x is well-founded.
3. $x\,;y$ is well-founded if x and y are well-founded.
4. σx is well-founded if x is well-founded.

Fact 52 Every hereditarily finite set is well-founded.

Fact 53 (Absence of Loops) Let x be well-founded. Then:

1. $x \notin x$.
2. If $y \in x$, then $x \notin y$.
3. If $z \in y$ and $y \in x$, then $x \notin z$.

Proof Each claim follows by induction on *WF x*. We show the first claim. Let *WF x*. We show $x \notin x$ by induction on *WF x*. The inductive hypothesis gives us $y \notin y$ for every $y \in x$. The claim follows. ∎

Fact 54 If x is well-founded, then $\sigma x \notin x$ and $\sigma x \neq x$.

Proof Follows from the absence of loops since $x \in \sigma x$. ∎

Since $x \subseteq \sigma x$ for every set x, we now know that the successor of a well-founded set x is strict superset of x.

Fact 55 The class of well-founded sets is not realizable.

Proof Suppose the set x realises the class of well-founded sets. Then every element of x is well-founded. Hence x is well-founded. Thus σx is well-founded. Thus $\sigma x \in x$. Contradiction. ∎

Exercise 56 Let $x \in y$ and x be well-founded. Prove $y \notin \sigma x$.

11 Separation and Intersection

We now postulate that every subset of a set exists. Put differently, we postulate that every subclass of a realizable class is realizable. To do so, we assume a function that, given a set x and a class p, yields the subset of x comprised by the elements of x that satisfy p.

$$\begin{aligned} sep\ x\ p &: set \to (set \to Prop) \to set \\ Sep &: \forall xpz.\ z \in sep\ x\ p \leftrightarrow z \in x \wedge pz \end{aligned}$$

Sets of the form $sep\ x\ p$ are called **separations**. We shall use the suggestive notations $x \cap p$ and $\{ z \in x \mid pz \}$ for a separation $sep\ x\ p$.

Fact 57 $x \cap p \subseteq x$ and $x \cap p \subseteq p$ and $\emptyset \cap p = \emptyset$.

Fact 58 $x \cap p \subseteq y$ if $x \subseteq y$.

Fact 59 $y \subseteq x \cap p$ if $y \subseteq x$ and $y \subseteq p$.

With separation we can define **binary intersection**:

$$x \cap y \ := \ \{ z \in x \mid z \in y \}$$

Fact 60 $z \in x \cap y$ if and only if $z \in x$ and $z \in y$.

Fact 61

1. $u \subseteq x \cap y$ if $u \subseteq x$ and $u \subseteq y$.
2. $x \cap y \subseteq u$ if $x \subseteq u$ or $y \subseteq u$.

We also define a **general intersection** operator:

$$\bigcap x \ := \ \{ z \in \bigcup x \mid \forall y \in x.\ z \in y \}$$

General intersection is dual to general union in that properties of unions appear in symmetric form as properties of intersections. There is a little asymmetry in that for most properties of an intersection $\bigcap x$ we need to require that x is inhabited. This is due to the fact that we do not have the dual of the empty set, which would be the universal set (which does not exist, see below). In mathematical texts one usually says that $\bigcap x$ is only defined if x is inhabited.

Fact 62 Let x be inhabited. Then:

1. $z \in \bigcap x$ if and only if z is an element of every element of x.
2. $u \subseteq \bigcap x$ if and only if u is a subset of every element of x.

Fact 63 $\bigcap x \subseteq u$ if u is an element of x.

Fact 64 (Greatest Low Bound Characterization)
Let x be inhabited. Then $\bigcap x = u$ if and only if the following conditions hold:
1. u is a subset of every element of x.
2. u is a superset of every set that is a subset of every element of x.

Fact 65 $\bigcap \emptyset = \emptyset$, $\bigcap \{x\} = x$, $\bigcap \{x, y\} = x \cap y$, and $\bigcap (x, y) = \{x\}$.

Fact 66 (Anti-Monotonicity) $\bigcap x \subseteq \bigcap y$ if $y \subseteq x$ and y is inhabited.

Fact 67 If $\bigcap x$ is inhabited, then x is inhabited.

Fact 68 $\bigcap x$ is transitive if every element of x is transitive.

Fact 69 $\bigcap x \subseteq \bigcup x$.

Exercise 70 Show that every superclass of an unrealizable class is unrealizable.

Exercise 71 Show that the class of all sets is unrealizable.

Exercise 72 Show that the class of finite sets is unrealizable.

Exercise 73 Show that there exists no set containing all its subsets.

Exercise 74 (Projections for Ordered Pair)
Recall that we know $\bigcup (x, y) = \{x, y\}$ and $\bigcap (x, y) = \{x\}$.
a) Define a function $\pi_1 : set \to set$ and prove $\pi_1 (x, y) = x$.
b) Define a function $\pi_2 : set \to set$ and prove $\pi_2 (x, y) = y$.
Hint: $\pi_2 u := \bigcup ((\bigcup u) \cap (\lambda z.\ u = (\pi_1 u, z)))$ does the job.

12 Infinity Axiom and Omega

With the current axioms we cannot proof that an infinite set exists. In fact, we cannot even prove that there exists a set containing all numerals. Thus we need an axiom providing for the existence of infinite sets. In its simplest form such an axiom says that a set containing all numerals exists. This is the so-called infinity axiom. We will work with a slightly stronger axiom asserting that the class of hereditarily finite sets is realizable.

$$\begin{aligned} hf &: set \\ Hf &: \forall z.\ z \in hf \leftrightarrow HF\ z \end{aligned}$$

We now define a subset ω of hf:

$$\omega := hf \cap Num$$

Fact 75 ω realizes the class of numerals.

Fact 76 ω is not finite.

Proof Assume ω is finite. Then there exists a list A of numerals such that $\omega = \mathit{fin}\ A$. By induction on A it follows that there exists a number n such that every element of A is an element of the numeral $\overline{n}$. Since $\omega = \mathit{fin}\ A$, it follows that $\overline{Sn} \in A$. Thus $\overline{Sn} \in \overline{n}$ and $Sn < n$. Contradiction. ∎

It is possible to characterize ω without using numbers and numerals. We call a set x **σ-closed** if $\emptyset \in x$ and $\sigma y \in x$ whenever $y \in x$.

Fact 77 ω is the least σ-closed set.

Fact 78 $\omega = \mathit{hf} \cap (\lambda x.$ every σ-closed set contains $x)$.

Fact 79 $\overline{n} \subseteq \omega$ for every number n.

Fact 80 ω is a transitive set.

Fact 81 $\bigcup \omega = \omega$.

13 Replacement, Cartesion Product, and Description

Replacement is a primitive operation that for a set u and a functional predicate $R : \mathit{set} \to \mathit{set} \to \mathit{Prop}$ yields the set $\{\, y \mid \exists x.\ x \in u \wedge Rxy \,\}$. Functionality of binary predicates is defined as one would expect:

$$\mathit{functional}\ R \ := \ \forall xyz.\ Rxy \to Rxz \to y = z$$

We axiomatize replacement as follows:

$$\begin{aligned} \mathit{rep} &: \mathit{set} \to (\mathit{set} \to \mathit{set} \to \mathit{Prop}) \to \mathit{set} \\ \mathit{Rep} &: \forall uRy.\ \mathit{functional}\ R \to (y \in \mathit{repl}\ u\ R \leftrightarrow \exists x \in u.\ Rxy) \end{aligned}$$

We call $\mathit{repl}\ u\ R$ the **range of R for u**. The requirement that R be functional is essential since otherwise replacement would assert the existence of non-existing sets (e.g., the universal set). The replacement axiom was formulated by Adolf Fraenkel [5] in 1922. The replacement axiom is needed to obtain certain sets whose existence cannot be proven in Zermelo's [11] initial axiomatization of set theory.

It is easy to see that separation is a special case of replacement.

Fact 82 $u \cap p = rep\ u\ (\lambda xy.\ x{=}y \wedge px)$.

Thus the axioms for separation are not needed.

With replacement and *hf* we can express the empty set.

Fact 83 $\emptyset = rep\ hf\ (\lambda xy.\bot)$.

Thus the axioms for the empty set may be omitted.

With replacement we can express sets of the form $\{\, Fx \mid x \in u \,\}$ where F is a function from sets to sets. We can now define the **cartesian product** operation for sets:

$$X \times Y \;:=\; \bigcup \{\, \{\, (x,y) \mid y \in Y \,\} \mid x \in X \,\}$$

Fact 84 $z \in X \times Y$ if and only if $z = (x,y)$ for some $x \in X$ and some $y \in Y$.

We can also define an operation $\delta : (set \to Prop) \to set$ known as **description**

$$\delta p \;:=\; \bigcup (rep\ \{\emptyset\}\ (\lambda xy.\ py))$$

that for a singleton class yields the set satisfying the class.

Fact 85 $\delta p = x$ if $\{x\}$ realizes p.

Fact 86 $p(\delta p)$ if p is a singleton class.

Exercise 87 Define a function $frep : set \to (set \to set) \to set$ and prove that $z \in frep\ u\ F$ if and only if $z = Fx$ for some $x \in u$.

Exercise 88 Replacement is inconsistent if the requirement that the predicate R be functional is dropped. Prove this claim by giving a relation R such that the class $\lambda y.\ \exists x{\in}u.\ Rxy$ is unrealizable for every inhabited set u.

Exercise 89 Show that replacement can express big unions of the form $\bigcup_{x \in u} Fx$.

14 Transitive Closure

With replacement we can define an operation *tc* that yields the transitive closure of a set. First note that $\bigcup x$ yields the set consisting of all elements of elements of x. By iterating the union operation on x we can get every element of the transitive closure of x. In fact, we have

$$tc\ x \;=\; x \cup (\bigcup x) \cup (\bigcup^2 x) \cup (\bigcup^3 x) \cup \;\cdots\; = \bigcup_{n \in \mathbf{N}} (\bigcup^n x)$$

We use replacement to define a union operator

$$\bigcup_{n \in \mathbf{N}} fn \;:=\; \bigcup f \;:=\; \bigcup (rep\ \omega\ (\lambda xy.\ \exists n.\ x = \overline{n} \wedge y = fn))$$

for a **number-indexed family** $f : \mathbf{N} \to set$ of sets and define the **transitive closure operator** as follows:

$$tc\ x \;:=\; \bigcup (\lambda n. \bigcup^n x)$$

Fact 90 $z \in \bigcup f$ if and only if $\exists n.\ z \in fn$.

Fact 91 $tc\ x$ is the least transitive superset of x.

Fact 92 $\bigcup f$ is transitive if every set fn is transitive.

Exercise 93 Draw the graph of the transitive closure of the singleton numeral for 3 and convince yourself that it is different from the von Neumann numeral for 3 (see also Exercise 40).

15 Power Sets

We assume a function that yields for every set its **power set**.

$$\mathcal{P} \;:\; set \to set$$
$$Power \;:\; \forall xz.\ z \in \mathcal{P}x \leftrightarrow z \subseteq x$$

Fact 94

1. $\mathcal{P}$ is monotone: $\mathcal{P}x \subseteq \mathcal{P}y$ if $x \subseteq y$.
2. $\mathcal{P}$ preserves transitivity: $\mathcal{P}x$ transitive if x is transitive.
3. $\mathcal{P}$ preserves well-foundedness: $\mathcal{P}x$ well-founded if x is well-founded.
4. Union undoes power: $\bigcup(\mathcal{P}x) = x$.
5. x is transitive if and only if $x \subseteq \mathcal{P}x$.
6. $\mathcal{P}x \notin x$.

Proof Claim (4) follows with Exercise 73. ∎

Theorem 95 (Cantor) Let u be a set and R be a functional predicate on sets. Then there exists a subset $v \subseteq u$ such that $v \notin rep\ u\ R$.

Proof Let $v := \{x \in u \mid \neg\exists y.\ Rxy \wedge x \in y\}$. Let $v \in rep\ u\ R$. Then Rxv for some $x \in u$. It easy to show that $x \in v$ if and only if $x \notin v$. Contradiction. ∎

Corollary 96 Every set has a subset that is not an element of the set.

Proof Cantor's theorem with $R := \lambda xy.\ x{=}y$. ∎

Note that with excluded middle Corollary 96 follows from Exercise 73.

16 Finite Power Universes

We consider the number-indexed family of sets obtained by iterating the power set operation on the empty set:

$$V_n := \mathcal{P}^n \emptyset$$

Fact 97

1. $V_0 \in V_1 \in V_2 \in V_3 \in \cdots$.
2. $V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$.

Proof Claim (1) follows from $x \in \mathcal{P}x$. Claim (2) follows by induction from $V_0 \subseteq V_1$ and the monotonicity of $\mathcal{P}$. ∎

Fact 98 V_n is transitive and well-founded.

Proof Holds since $\mathcal{P}$ preserves transitivity and well-foundedness. ∎

Fact 99 $V_{n+k} \notin V_n$.

Fact 100 (Closure Properties)

1. If $x \in V_n$ and $y \in V_n$, then $\{x, y\} \in V_{n+1}$.
2. If $x \in V_n$, then $\bigcup x \in V_n$.
3. If $x \in V_n$, then $\mathcal{P}x \in V_{n+1}$.

We define $V_\omega := \bigcup_{n \in \mathbf{N}} V_n$.

Fact 101

1. $V_n \in V_\omega$ and $V_n \subseteq V_\omega$.
2. V_ω is transitive and well-founded.
3. V_ω is closed under taking unordered pairs, unions, and power sets.
4. If $x \subseteq V_\omega$ is finite, then $x \in V_\omega$.

Fact 102 $HF \subseteq V_\omega$.

17 Excluded Middle

Recall the definition of excluded middle:

$$XM := \forall P.\, P \vee \neg P$$

No result shown so far required the use of excluded middle. Nevertheless, many familiar mathematical facts about sets require the use of excluded middle and are in fact equivalent to excluded middle.

First, we observe that separation introduces an intimate connection between propositions and set membership.

Lemma 103 Let P be a proposition. Then P if and only if $\{\emptyset\} \cap (\lambda x.P)$ is inhabited.

Fact 104 The following propositions are equivalent to *XM*.

1. Every set is either inhabited or not inhabited.
2. For every set $x \subseteq \{\emptyset\}$ either $\emptyset \in x$ or $\emptyset \notin x$.
3. Every subset of $\{\emptyset\}$ is finite.

Proof The claims follow with Lemma 103. ∎

Fact 105 The following propositions are equivalent to *XM*.

1. $\mathcal{P}\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$.
2. $\mathcal{P}\{\emptyset\}$ is hereditarily finite.

Proof Let $HF(\mathcal{P}\{\emptyset\})$. We show *XM*. By Fact 104 (2) it suffices to show that every subset of $\{\emptyset\}$ is finite. Let $x \subseteq \{\emptyset\}$. Then $x \in \mathcal{P}\{\emptyset\}$. Hence x is a member of hereditarily finite set. Thus x is finite.

The rest is straightforward. ∎

We now state a number of results whose proofs use *XM*. We will state the use of *XM* explicitly in the head of a lemma. We start with a lemma that is the key for the following results.

Lemma 106 (XM) If $x \subseteq y \cup \{z\}$, then either $x \subseteq y$ or $x = \{u \in x \mid u \neq z\} \cup \{z\}$.

Proof Let $x \subseteq y \cup \{z\}$. Case analysis with *XM* for $z \in x$.

1. $z \in x$. We show $x = \{u \in x \mid u \neq z\} \cup \{z\}$ using extensionality. The proof from left to right uses *XM* for $u = z$. The other direction is straightforward.
2. $z \notin x$. Then $x \subseteq y$. ∎

Fact 107 (XM) Every subset of a finite set is finite.

Proof Let $x \subseteq \mathit{fin}\, A$. We show that x is finite by induction on the list A. If $A = \mathit{nil}$, then $x = \emptyset$ and hence finite. Let $A = y :: B$. Then $x \subseteq \mathit{fin}\, B \cup \{y\}$. Case analysis with Lemma 106.

1. $x \subseteq fin\ B$. Then the claim follows by the inductive hypothesis.
2. $x = \{ z \in x \mid z \neq y \} \cup \{y\}$. It suffices to show that $\{ z \in x \mid z \neq y \}$ is finite. This follows with the inductive hypothesis since $\{ z \in x \mid z \neq y \} \subseteq fin\ B$. ■

Fact 108 (XM) V_ω is infinite.

Proof Follows with Fact 107 since $\omega \subseteq V_\omega$ is infinite. ■

We now come to an important classical fact about power sets.

Fact 109 (XM) $\mathcal{P}(x \cup \{y\}) = \mathcal{P}x \cup \{ z \cup \{y\} \mid z \in \mathcal{P}x \}$.

Proof The direction from left to right follows with Lemma 106. No further use of *XM* is required. ■

Using this fact we can verify that the power set of a finite set can be computed with the power list function.

Fact 110 (XM) $\mathcal{P}(fin\ A) = fin\ (map\ fin\ (power\ A))$.

This fact has remarkable consequences.

Fact 111 (XM) The power set of a finite set is finite.

Fact 112 (XM) The power set of a hereditarily finite set is hereditarily finite.

Fact 113 (XM) $V_\omega \subseteq HF$.

Theorem 114 (XM) V_ω realizes *HF*.

Proof Follows with Fact 113 and Fact 102. ■

Note that in the chain of reasoning leading to Theorem 114 the only explicit use of *XM* is in the proof of Lemma 106.

18 Ordinals

Recall the most important properties of transitive sets:

1. x is transitive if and only if every element of x is a subset of x.
2. If every element of x is transitive, then $\bigcup x$ is transitive.
3. If every element of x is transitive, then $\bigcap x$ is transitive.
4. If x is transitive, then σx is transitive.

5. If x is transitive, then $\bigcup(\sigma x) = x$ and $\bigcup x \subseteq x$.
6. σ is injective on transitive sets.

A particularly important class of sets are the **hereditarily transitive sets**:

$$\frac{\mathit{trans}\ x \qquad x \subseteq HT}{HT\ x}$$

Hereditarily transitive sets are well-founded sets whose graph is transitive.

Fact 115 The following statements are equivalent:

1. x is hereditarily transitive.
2. x is transitive and every element of x is hereditarily transitive.
3. x is well-founded and transitive, and every element of x is transitive.

Hereditarily transitive sets are better known as **ordinals** and this will be the name we will use.

Fact 116 (Closure Properties)

1. If x is a set of ordinals, then $\bigcup x$ and $\bigcap x$ are ordinals.
2. If x is an ordinal, then σx, $\bigcup x$, and $\bigcap x$ are ordinals.

Fact 117 The numerals and ω are ordinals.

Fact 118

1. If x is a set of ordinals, then $\sigma(\bigcup x)$ is an ordinal that is not in x.
2. The class of ordinals is not realizable.

Following a common convention, the letter α, β, and γ will be reserved for ordinals.

Fact 119 (Linearity, XM) Let α and β be ordinals. Then $\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$.

Proof By nested induction on the ordinals α and β. Since we have XM, it suffices to show that $\alpha = \beta$ if $\alpha \notin \beta$ and $\beta \notin \alpha$. Let $\alpha \notin \beta$ and $\beta \notin \alpha$. We show $\alpha = \beta$ using extensionality.

Let $\gamma \in \alpha$. We show $\gamma \in \beta$. By the inductive hypothesis for α we have one of the following cases:

1. $\gamma \in \beta$. Then we are done.
2. $\gamma = \beta$. Then $\beta \in \alpha$. Contradiction.
3. $\beta \in \gamma$. Then $\beta \in \alpha$ since α is transitive. Contradiction.

The other direction is analogous. ∎

We now know that the ordinals are well-ordered by membership where $\alpha \in \beta$ means $\alpha < \beta$. Well-ordering means that we have a linear order not allowing for infinite descending chains. The absence of infinite descending chains follows from the fact that ordinals are well-founded. It is clear that $\emptyset$ is the least ordinal.

Fact 120 (XM) $\alpha \subseteq \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$.

Proof Let $\alpha \subseteq \beta$. Because of linearity it suffices to show that $\beta \in \alpha$ is contradictory. This is the case since it implies $\alpha \in \alpha$ and α is well-founded. The other direction is obvious. ∎

If $\alpha \in \beta$, then $\alpha \neq \beta$ since α is well-founded. We are now justified to write $\alpha < \beta$ for $\alpha \in \beta$ and $\alpha \leq \beta$ for $\alpha \subseteq \beta$.

Exercise 121 (XM) Show that every inhabited ordinal contains $\emptyset$.

Exercise 122 (XM) Show that an ordinal is finite if and only if it is a numeral.

19 Successor and Limit Ordinals

The ordinals are partitioned into sucessor ordinals and limit ordinals, which are defined as follows:

- α is a **successor ordinal** if $\alpha = \sigma\beta$ for some ordinal β.
- α is a **limit ordinal** if $\alpha = \bigcup \alpha$.

We will show that the every ordinal is either a successor or a limit ordinal and that limit ordinals and sucessor ordinals are disjoint.

Fact 123 $\bigcup \alpha \subseteq \alpha$ and $\alpha \notin \bigcup \alpha$.

Proof Follows by transitivity and well-foundedness of α. ∎

Fact 124 (XM) Either $\bigcup \alpha = \alpha$ or $\bigcup \alpha \in \alpha$.

Proof Follows with linearity from Fact 123. ∎

We now show that an ordinal α is a successor ordinal if and only if $\bigcup \alpha \in \alpha$.

Fact 125 (XM) $\bigcup \alpha \in \alpha$ if and only if $\alpha = \sigma(\bigcup \alpha)$.

Proof The direction from right to left is obvious. For the other direction, let $\bigcup \alpha \in \alpha$. We show $\alpha = \sigma(\bigcup \alpha)$ by extensionality.

Let $\beta \in \alpha$. Then $\beta \subseteq \bigcup \alpha$. By linearity we have either $\beta \in \bigcup \alpha$ or $\beta = \bigcup \alpha$. Hence $\beta \in \sigma(\bigcup \alpha)$.

Let $\beta \in \sigma(\bigcup \alpha)$. Then either $\beta \in \bigcup \alpha$ or $\beta = \bigcup \alpha$. Hence $\beta \in \alpha$ since $\bigcup \alpha \in \alpha$. ∎

Fact 126 (Partition into Successors and Limits, XM)

1. Successor ordinals and limit ordinals are disjoint.
2. Every ordinal is either a successor ordinal or a limit ordinal.

Proof Follows with Facts 124 and 125. ∎

We can now characterize ordinals as sets that can be obtained with union and successor.

$$\frac{x \subseteq ON'}{ON'(\bigcup x)} \qquad\qquad \frac{ON'\ x}{ON'(\sigma x)}$$

Fact 127 (XM) $ON\ x$ if and only if $ON'\ x$.

Proof Each direction follows by induction. The direction from ON to ON' exploits that an ordinal is either a limit or a successor. The direction from ON' to ON exploits that ON is closed under taking unions and successors. ∎

Fact 128 (Successor) Let α be an ordinal.

1. $\alpha < \sigma\alpha$.
2. Union undoes successor: $\bigcup(\sigma\alpha) = \alpha$.
3. Tightness: There exists no set x such that $\alpha < x < \sigma\alpha$.

Exercise 129 (XM) Show that ω is the least inhabited limit ordinal.

Exercise 130 (XM) Show that the following statements are equivalent for every ordinal α.

a) α is a successor ordinal.

b) $\bigcup\alpha \in \alpha$.

c) α has a greatest element.

Exercise 131 Write and verify a function that for every ordinal α yields the least limit ordinal γ such that $\alpha \leq \gamma$.

20 First-Order Characterization of Ordinals

Can we define the class of ordinals without using an inductive definition and by just quantifying over sets? The answer is yes and will lead us to the standard definition of ordinals in the literature. We speak of the first-order characterization of ordinals, where by first-order we mean that the characterization does not use inductive definitions and just quantifies over sets.

We know that an ordinal is a set such that membership is a well-ordering on the elements of the ordinal. A well-ordering is a linear ordering without infinite descending chains. The standard example of a well-ordering is the ordering on the natural numbers. In our development, the natural numbers appear as the elements of the ordinal ω and the well-ordering of the natural numbers appears as the membership relation restricted to the elements of ω. It will turn out that the ordinals are exactly those sets whose elements are well-ordered by the membership relation.

For the formal development we do not want to use the notion of an infinite descending chain. It turns out that a set does not have infinite descending chains if and only if every inhabited subset has a minimal element. Since the ordering we are considering is membership, an element of a set is minimal if it does not have a common element with the set. Recall that two sets are called disjoint if they that do not have a common element. This motivates the following definitions.

- Two sets x and y are **disjoint** if there is no set z such that z is an element of both x and y.
- A set y is a **minimal element** of a set x if $y \in x$ and y and x are disjoint.
- A set x is **regular** if in case it is inhabited it has a minimal element.
- A set x is **subset regular** if every subset of x is regular.
- A set x is **serial** if it is inhabited and for every element $y \in x$ there exists an element $z \in y$ such that $z \in x$.

Note that every infinite descending chain can be represented as a serial set. Moreover, a transitive set admits an infinite chain if it contains a serial set.

Fact 132 No set is both regular and serial.

Fact 133 (XM) A set is serial if and only if it is not regular.

Proof Rewriting with double negation and de Morgan laws. ∎

Fact 134 (XM) Every well-founded set is regular.

Proof Let x be well-founded. If x is inhabited, we have a well-founded set $y \in x$. We prove by induction on the well-foundedness of y that x has an element that is disjoint with x. We use excluded middle. If y is disjoint with x, we are done. Otherwise, we have an element $z \in y$ such that $z \in x$. Since z is well-founded, the claim follows by the inductive hypothesis. ∎

Corollary 135 (XM) Every well-founded set is subset regular.

Theorem 136 (XM) Let x be an inhabited set of ordinals. Then $\bigcap x$ is a minimal element of x.

Proof Since x is well-founded and inhabited, we have by Fact 134 a set $y \in x$ such that y and x are disjoint. We show $\bigcap x = y$ using the greatest lower bound criterion.

1. Let $z \in x$. We show $y \subseteq z$. By linearity it suffices to show that $z \notin y$. This is the case since otherwise z would be common element of x and y, which we know are disjoint.
2. Let v be subset of every element of x. We have to show that v is a subset of y. This is the case since $y \in x$. ∎

Fact 137 (XM) Every non-well-founded set has a non-well-founded element.

Proof Let x be a non-well-founded set. We show the claim by contradiction. Suppose every element of x is well-founded. Then x is well-founded. Contradiction. ∎

Fact 138 (XM) Let x be a transitive and non-well-founded set. Then the subset $\{\, y \in x \mid \neg \mathit{WF}\ y \,\}$ is serial.

Proof Follows with Fact 137. ∎

Fact 139 (XM) Every transitive and subset regular set is well-founded.

Proof Let x be transitive and subset regular. We show the claim by contradiction. Suppose x is not well-founded. Then Fact 138 gives us a serial subset of x. Contradiction since x is subset regular and serial sets are not regular. ∎

A set u is **linear** if for every $x \in u$ and every $y \in u$ either $x \in y$ or $x = y$ or $y \in x$. We now have everything on the table so that we can the usual first-order characterization of ordinals.

Theorem 140 (First-Order Characterization of Ordinals, XM)
A set is an ordinal if and only if it is transitive, linear, and subset regular.

Proof Let x be an ordinal. Then x is transitive and linear. Since x is also well-founded, we know by Corollary 135 that x is subset regular.

Let x be a transitive, linear, and subset regular set. The x is well-founded by Fact 139. By Fact 115 it suffices to show that every element of x is transitive. Let $u \in z \in y \in x$. We have to show $u \in y$. Since x is transitive, $u \in x$. Since x is linear, we have either $u \in y$ or $u = y$ or $y \in u$. The second and third case contradict the well-foundedness of x. Thus $u \in y$. ∎

Fact 141 A set is well-founded if and only if its transitive closure is well-founded.

Proof The direction from right to left follows from the fact that every set is a subset of its transitive closure. For the other direction assume that x is well-founded. It suffices to show that $\bigcup^n(x)$ is well-founded for every n, which follows by induction on n since x is well-founded. ■

Theorem 142 (First-Order Characterization of Well-Foundedness, XM)
A set is well-founded if and only if its transitive closure is subset regular.

Proof Follows by Facts 141, Corollary 135 and Fact 139. ■

We already mentioned that the standard set theory ZF comes with an axiom saying that every set is well-founded. This can be expressed with a first-order axiom that says that all sets are regular.

Theorem 143 (XM) All sets are well-founded if and only if all sets are regular.

Proof The direction from left to right follows with Fact 134. We prove the other direction by contradiction. Assume every set is regular and that there is a set that is not well-founded. By Fact 141 we know that there is a transitive set that is not well-founded. By Fact 138 we have a serial set. By Fact 132 this contradicts the assumption that every set is regular. ■

Exercise 144 (XM) Let x be an inhabited set of ordinals. Show that $\bigcap x$ is the least element of x.

21 Cumulative Hierarchy

We define the class *Cum* of **cumulative sets** as follows:

$$\frac{x \subseteq Cum}{Cum\,(\bigcup x)} \qquad\qquad \frac{Cum\,x}{Cum\,(\mathcal{P}x)}$$

Note that the definition of the class *Cum* agrees with the definition of the class ON' except that taking successors is replaced by taking powers. In both cases the defining rules preserve transitivity and well-foundedness.

Fact 145 V_n and V_ω are cumulative.

Fact 146 Cumulative sets are transitive and well-founded.

Fact 147 The class of cumulative sets is not realizable.

We end the lecture notes with two challenge problems, which are educated conjectures the professor has not proven yet.

Conjecture 148 (Linearity, XM)
Let x and y be cumulative sets. Then $x \in y \vee x = y \vee y \in x$.

Given the conjecture, one can show that every inhabited subclass of *Cum* contains a unique least element. This provides for the construction of a choice function for classes of cumulative sets.

Conjecture 149 (Choice Function, XM)
There is a function $F : (set \to Prop) \to set$ such $p(Fp)$ for every inhabited subclass $p \subseteq Cum$.

Corollary 150 (XM)
Every well-founded set is an element of some cumulative set.

Proof By induction on *WF* using a choice function as specified by Conjecture 149. ∎

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