Set Theory in Type Theory

Lecture Notes

Gert Smolka
Saarland University
January 12, 2017

1 Introduction

Modern mathematics takes sets as universal data structure and represents all mathematical objects (e.g., numbers, relations, functions) as sets. Sets and their properties are made precise by means of an axiomatisation. It suffices to consider pure sets whose elements are sets.

We study a basic axiomatic set theory in constructive type theory assuming excluded middle. There are only few direct uses of excluded middle, but results derived with excluded middle are essential when reasoning about infinite sets.

Constructive type theory is surprisingly well-suited for the development of axiomatic set theory. Important classes of sets such as (hereditarily) finite sets, well-founded sets, ordinals, and the stages of the cumulative hierarchy can be captured naturally with inductive predicates providing useful induction principles. We depart substantially from conventional developments of set theory, where inductive characterisations and inductive proof methods are rarely used.

There is an accompanying Coq development following the mathematical presentation in the notes. It turns out that axiomatic set theory is an excellent case study for the use of interactive theorem provers. We have written the formal proofs in the Coq development so that they are close to the informal proofs in the notes. Often, the Coq proofs are shorter than the informal proofs.

Our development of axiomatic set theory in type theory should provide a quick start into the foundations of mathematics for both computer scientists and mathematicians. In contrast to a traditional mathematical introduction, our presentation has the substantial advantage that the underlying logic providing for the axiomatisation and the theorems is explicit and computer-implemented.

The standard axiomatic set theory is known as ZFC (for Zermelo, Fraenkel, and axiom of choice). ZFC is usually presented in first-order logic where only sets are first-class objects (i.e., quantification is restricted to sets). The first-order presentation is unfortunate in that it forces a low-level coding of important set-theoretic ideas that can be expressed naturally in a higher-order type theory with inductive predicates.

2 Sets and Classes

The basic setup of our axiomatic set theories is straightforward: We assume a type set and a binary predicate $\in$ on set modelling set membership:

$$\begin{aligned} \mathsf{set} &: \mathrm{Type} \\ \in &: \mathsf{set} \to \mathsf{set} \to \mathbb{P} \end{aligned}$$

Note that the type of the membership predicate ensures that the elements of sets are sets. The letters x, y, z, a, b, c will range over sets in the following.

A **class** (of sets) is a unary predicate on sets. The letters p and q will range over classes. A set x **agrees** with a class p if the elements of x are exactly the sets satisfying p:

$$\text{agree}\ x\ p \ := \ \forall z.\ z \in x \leftrightarrow pz$$

We call a class **small** if it agrees with some set, and **large** if it is not small. There is a wellknown large class discovered by Bertrand Russell [10].

Fact 1 (Russell) $\lambda x.x \notin x$ is large.

Proof Suppose $\lambda x.x \notin x$ agrees with y. T hen $y \in y \leftrightarrow y \notin y$. Contradiction. ▪

We can reformulate Russel's fact such that it doesn't use the notion of a class.

Fact 2 (Russell)
There is no set containing exactly those sets that do not contain themselves.

Russell's fact is known as *Russell's paradox*. Russell's paradox made it clear that Cantor's naive set theory, which did not make rigorous assumptions about the existence of sets, needed a proper mathematic foundation.

We define **set inclusion** as follows:

$$x \subseteq y \ := \ \forall z.\ z \in x \to x \in y$$

Following the usual convention, we call x a **subset** of y and y a **superset** of x if $x \subseteq y$. We also introduce a notation for **proper subsets**:

$$x \subset y \ := \ x \subseteq y \wedge x \neq y$$

A basic assumption of set theory is **extensionality**: Two sets are identical if they have the same elements. We assume extensionality with the following axiom:

$$\text{Ext} \ : \ \forall xy.\ x \subseteq y \to y \subseteq x \to x = y$$

Extensionality says that inclusion is antisymmetric. Since inclusion is reflexive and transitive by definition, inclusion is a partial order on sets. The inclusion ordering of sets is an essential notion in set theory.

Fact 3 Set inclusion is a partial order on sets.

Recall that in a partial order we can talk about least and greatest elements satisfying a certain property. Given a property, there is at most one least and at most one greatest element satisfying the property.

Exercise 4 Show that a class agrees with at most one set.

3 Empty Set

We need axioms to establish the existence of sets. With the assumptions so far, we cannot prove that a set exists. Given Russel's paradox, we need to be careful when we assume the existence of sets. Axiomatic set theory asserts the existence of certain sets with carefully chosen axioms.

We start with two axioms saying that an empty set exists.

$$\emptyset : \text{set}$$
$$\text{Eset} : \forall x.\, x \notin \emptyset$$

Fact 5 (Uniqueness) $\emptyset$ is the only set containing no elements.

Proof Follows with extensionality. ∎

The precise formulation of the uniqueness fact is $\forall x.\ (\forall z.\ z \notin x) \to x = \emptyset$. Make sure that you can switch between informal and formal formulations easily. Full proofs work on formal formulations. Informal formulations matter for humans since they make it possible to ignore details and concentrate on essential aspects.

Clearly, the empty set is the least set (with respect to inclusion).

Fact 6 $\emptyset \subseteq x$. Thus $x \subseteq \emptyset \to x = \emptyset$.

A set is **inhabited** if it has an element.

$$\text{inhab } x := \exists z.\, z \in x$$

Fact 7 A set x is not inhabited if and only if $x = \emptyset$.

That a set is inhabited if it is different from $\emptyset$ can be shown if we assume excluded middle. For now, we will not assume excluded middle.

4 Adjunction

We now assume an operation $x.y$ called **adjunction** that yields the least superset of y containing x. If $x \in y$, then $x.y = x$. Otherwise, $x.y$ is the set obtained from y by adding the additional element x. The axioms for adjunction are straightforward:

$$\_.\_ : \text{set} \to \text{set} \to \text{set}$$
$$\text{Adj} : \forall xyz.\ z \in x.y \leftrightarrow z = x \vee z \in y$$

We adopt the convention that $a.b.x$ is read as $a.(b.x)$. We can now define the usual notation for finite sets:

$$\{x_1, \ldots, x_n\} \;:=\; x_1.\;\ldots\;.x_n.\emptyset$$

With $\emptyset$ and adjunction we can represent every finite set whose elements are finite sets. This informal observation will be made precise latter.

There is a certain similarity between finite sets and lists. We may say that adjunction takes the role of cons, and that the empty set takes the role of the empty list.

We list some prominent facts about adjunction. All of them have straightforward proofs. Some of them will be used frequently in further proofs, often tacitly.

Fact 8 (Cancel and Swap) $a.a.x = a.x$ and $a.b.x = b.a.x$.

Fact 9 (Discrimination) $a.x \neq \emptyset$ and $a.x \notin \emptyset$.

Fact 10 (Membership) $z \in x \;\leftrightarrow\; z.x = x$.

Fact 11 (Injectivity) $a.x = b.y \;\rightarrow\; a = b \vee a \in y$.

Fact 12 (Inclusion) $a.x \subseteq y \;\leftrightarrow\; a \in y \wedge x \subseteq y$.

Note that Fact 10 characterises membership with equality and adjunction.

A set containing exactly one element is called a **singleton**.

Fact 13 (Singletons)

1. $a \in \{b\} \rightarrow a = b$.
2. $\{a\} = \{b\} \rightarrow a = b$.
3. $x = \{a\} \rightarrow a \in x$.
4. $\{a, b\} = \{c\} \rightarrow a = c \wedge b = c$.

Fact 14 (Injectivity of unordered pairs)
$\{x, y\} = \{a, b\} \rightarrow x = a \wedge y = b \vee x = b \wedge y = a$.

Proof The assumption entails $x = a \vee x = b$ for x and similar disjunctions for y, a, and b. This yields 16 trivial cases. ∎

Ordered pairs $\langle x, y\rangle$ consisting of two sets x and y can be represented as sets. This matters since set theory represents functions and relations as sets of ordered pairs. Following Kuratowski [8], we represent **ordered pairs** as follows:

$$\langle \mathrm{x}, \mathrm{y}\rangle \;:=\; \{\{x\}, \{x, y\}\}$$

This representation makes sense since it is injective, that is, different pairs are represented as different sets.

Fact 15 (Injectivity of ordered pairs)
If $\langle x, y\rangle = \langle a, b\rangle$, then $x = a$ and $y = b$.

Proof Follows with the injectivity laws for singletons and unordered pairs. ∎

5 Numerals

The set theory we have axiomatised so far can represent natural numbers. The standard representation is due to John von Neumann [9] and represents a number as the set of all smaller number. Following this idea, zero must be represented as the empty set. We call the sets representing the numbers numerals. Here are the first four numerals:

$$\begin{array}{l} \emptyset \\ \{\emptyset\} \\ \{\{\emptyset\}, \emptyset\} \\ \{\{\{\emptyset\}, \emptyset\}, \{\emptyset\}, \emptyset\} \end{array}$$

The **successor function** generating the numerals starting from $\emptyset$ is self-adjunction:

$$\sigma x \;:=\; x.x$$

We define the class Num of **numerals** inductively:

$$\frac{}{\text{Num}\,\emptyset} \qquad\qquad \frac{\text{Num}\,x}{\text{Num}\,(\sigma x)}$$

Fact 16 (Closedness) Every element of a numeral is a numeral.

Proof By numeral induction. ∎

There is the difficulty that we cannot show that the successor function is injective on all sets, given the axioms assumed so far. However, the successor function is injective on a superclass of Num, the class of transitive sets. A set x is **transitive** if every element of x is a subset of x (i.e., $y \subseteq x$ for every $y \in x$). Note that a set x is transitive if and only if every element of every element of x is an element of x. The name transitive set is maybe best explained with the equivalence

$$\text{transitive } z \;\leftrightarrow\; \forall xy.\, x \in y \to y \in z \to x \in z$$

Fact 17 (Transitivity)
1. $\emptyset$ is transitive.

2. σx is transitive if x is transitive.
3. Every numeral is transitive.

Fact 18 (Injectivity)

1. Let y be transitive and $\sigma x \subseteq \sigma y$. Then $x \subseteq y$.
2. Let x and y be transitive and $\sigma x = \sigma y$. Then $x = y$.

Proof Let x and y be transitive sets and $\sigma x = \sigma y$. Because of extensionality and symmetry it suffices to show $\sigma x \subseteq \sigma y$. This follows with the injectivity property for adjunction (Fact 11). ∎

We now know that the class of numerals satisfies the Peano axioms.[1] Next we explore the canonical ordering of numerals. The construction of numerals suggests that $x \in y$ corresponds to $m < n$ and $x \subseteq y$ corresponds to $x \leq y$.

Fact 19 (Strictness) Let x be a numeral. Then $x \notin x$.

Proof By numeral induction using Fact 17. ∎

Fact 20 (Zero) Let x be a numeral. Then $x = \emptyset \vee \emptyset \in x$.

Proof By numeral induction on x. ∎

Fact 21 (Monotonicity) Let y be a numeral and $x \in y$. Then $\sigma x \in \sigma y$.

Proof By numeral induction on y. ∎

Fact 22 (Trichotomy) Let x and y be numerals. Then $x \in y \vee x = y \vee y \in x$.

Proof By numeral induction on x using Facts 20 and 21. ∎

Fact 23 (Linearity) Let x and y be numerals. Then:

1. $x \subseteq y$ or $\sigma y \subseteq x$.
2. $x \subseteq y$ or $y \in x$.
3. $x \subseteq y$ or $y \subseteq x$.

Proof Follows with trichotomy and transitivity. ∎

Fact 24 (Cumulativity) Let x and y be numerals. Then $x \in y \leftrightarrow x \subseteq y \wedge x \neq y$.

Proof Follows with linearity, transitivity, and strictness. ∎

[1] There are three Peano axioms: injective successor function, all successors disjoint from zero, and the induction principle.

Fact 25 (Monotonicity) Let y be a numeral and $x \subseteq y$. Then $\sigma x \subseteq \sigma y$.

Proof Follows with Facts 24 and 19. ∎

Exercise 26 Show that no numeral agrees with the class of numerals.

Exercise 27 Let $x \in y$ and let either x or y be a numeral. Show $y \notin x$.

Exercise 28 *Singleton numerals* are defined as follows: $\tilde{0} := \emptyset$, $\widetilde{Sn} := \{\tilde{n}\}$. Show that $\tilde{n}$ is injective. Zermelo represented numbers as singleton numerals [13].

Exercise 29 We map numbers to numerals as follows:

$$\overline{n} \ := \ \sigma^n \emptyset$$

Prove the following statements.

a) A set x is a numeral if and only if there is a number n such that $x = \overline{n}$.
b) If $m < n$, then $\overline{m} \in \overline{n}$.
c) If $x \in \overline{n}$, then $x = \overline{m}$ for some $m < n$.
d) $m = n$ if and only if $\overline{m} = \overline{n}$.
e) $m < n$ if and only if $\overline{m} \in \overline{n}$.
f) $m \leq n$ if and only if $\overline{m} \subseteq \overline{n}$.

6 Finite Sets

We define the class Fin of **finite sets** inductively:

$$\frac{}{\text{Fin}\,\emptyset} \qquad\qquad \frac{\text{Fin}\,y}{\text{Fin}\,(x.y)}$$

We call a set **infinite** if it is not finite.

Fact 30 Every numeral is a finite set.

Fact 31 Every finite set is either empty or inhabited.

A **chain** is a set x such that for all $a, b \in x$ either $a \subseteq b$ or $b \subseteq a$. A set u is a **greatest element** of a set x if $u \in x$ and $y \subseteq u$ for every $y \in x$.

Fact 32 Every inhabited finite chain has a greatest element.

Proof Let x be an inhabited finite chain. The existence of a greatest element follows by Fin induction on x. ∎

Exercise 33 Show that every set that agrees with Num is infinite.

7 Hereditarily Finite Sets

Given an infinite set x, the set $\{x\}$ is finite. Thus the class of finite sets is not closed in a set theory with infinite sets. We now define a maximal closed class of finite sets.

We define the class HF of **hereditarily finite sets** inductively:

$$\frac{}{\mathrm{HF}\,\emptyset} \qquad \frac{\mathrm{HF}\,x \qquad \mathrm{HF}\,y}{\mathrm{HF}\,(x.y)}$$

Fact 34 (Closedness) Every element of an HF set is an HF set.

Proof Let $x \in y$ and y be an HF set. That x is an HF set follows by HF induction on y. ■

Fact 35 HF sets are finite sets.

Fact 36 Numerals are HF sets.

A proposition P is **propositionally decidable** if $P \vee \neg P$. Assuming excluded middle amounts to assuming that every proposition is propositionally decidable. Interestingly, one can show that membership, inclusion, and equality of HF sets are propositionally decidable without assuming excluded middle. The proof is a bit involved. We will use the notation $\mathrm{xm}\,P := P \vee \neg P$.

Lemma 37

1. $\mathrm{xm}\,(z = a) \to \mathrm{xm}\,(z \in x) \to \mathrm{xm}\,(z \in a.x)$.
2. $\mathrm{xm}\,(a \in y) \to \mathrm{xm}\,(x \subseteq y) \to \mathrm{xm}\,(a.x \subseteq y)$.
3. $\mathrm{xm}\,(x \subseteq y) \to \mathrm{xm}\,(y \subseteq x) \to \mathrm{xm}\,(x = y)$.

Lemma 38 $\mathrm{HF}\,x \to \mathrm{xm}\,(\emptyset \in x)$.

Lemma 39 $\mathrm{HF}\,x \to \mathrm{HF}\,y \to \mathrm{xm}\,(x \in y) \wedge \mathrm{xm}\,(y \in x) \wedge \mathrm{xm}\,(x \subseteq y) \wedge \mathrm{xm}\,(y \subseteq x)$.

Proof By nested induction on $\mathrm{HF}\,x$ and $\mathrm{HF}\,y$ using the two preceding lemmas. For the induction to go through, it is essential that the four claims are shown together. ■

Theorem 40 Membership, inclusion, and equality of HF sets are propositionally decidable.

Is HF small? Our axioms so far do not decide this question. We could have the axiom that every set is an HF set, which gives us a finite set theory where HF is large. ZFC goes the other way and has axioms that ensure HF is small.

Exercise 41 Show that HF sets are closed under taking ordered pairs.

Exercise 42 Let pz be propositionally decidable for every HF set z. Prove that the propositions $\forall z \in x.\ pz$ and $\exists z \in x.\ pz$ are propositionally decidable for every HF set x.

Exercise 43 Prove the following.

a) For every HF set x there exists a list A of HF sets such that $z \in x \leftrightarrow z \in A$ for every set z.
b) For every list A of HF sets there exists an HF set x such that $z \in x \leftrightarrow z \in A$ for every set z.

8 Well-Founded Sets

Given a set, one can descent to an element of the set. A recursive descent terminates once the empty set is reached.

$$\emptyset = x_n \in \cdots \in x_1 \in x_0$$

Sets for which recursive descent always terminates are called *well-founded*. A set containing itself as element is clearly not well-founded. Given the inductive definition of HF sets, it is clear that every HF set is well-founded. With the usual axioms of set theory it is not possible to prove the existence of a non-wellfounded set. In fact, ZFC has an axiom saying that every set is well-founded (regularity axiom).

We shall use the notation

$$x \subseteq p \ := \ \forall z.\ z \in x \to px$$

We define the class WF of **well-founded sets** inductively with a single rule:

$$\frac{x \subseteq \mathrm{WF}}{\mathrm{WF}\, x}$$

It says that every set of well-founded sets is well-founded.

Fact 44 A set is well-founded if and only if each of its elements is well-founded.

Fact 45 Every subset of a well-founded set is well-founded.

Fact 46 The empty set is well-founded. Moreover, the class of well-founded sets is closed under adjunction. Thus every HF set is well-founded.

The inductive predicate WF provides an induction principle that will be useful: To show that every well-founded set satisfies a property p, it suffices to show that a set of well-founded sets satisfies p if all its elements satisfy p. WF induction is known as **epsilon induction** in set theory.

Fact 47 (Absence of Loops) Let x be well-founded. Then:

1. $x \notin x$.
2. If $y \in x$, then $x \notin y$.
3. If $z \in y$ and $y \in x$, then $x \notin z$.

Proof Each claim follows by WF induction on x. We show the first claim. Let x be well-founded. We show $x \notin x$ by WF induction on x. The inductive hypothesis gives us $y \notin y$ for every $y \in x$. The claim follows. ∎

Fact 48 The class of well-founded sets is large.

Proof Suppose the set x agrees with the class of well-founded sets. The x is well-founded since every element of x is wellfounded. Hence $x \in x$, which is contradictory. ∎

Fact 49 (Injectivity)

1. Let y be well-founded and $\sigma x \subseteq \sigma y$. Then $x \subseteq y$.
2. Let x and y be well-founded and $\sigma x = \sigma y$. Then $x = y$.

Proof Let y be well-founded and $\sigma x \subseteq \sigma y$. We show $x \subseteq y$. We have $x = y$ or $x \in y$ since $x \in \sigma y$. Let $x \in y$ and $z \in x$. We show $z \in y$. We have $z = y$ or $z \in y$ since $z \in \sigma y$. The case $z = y$ is contradictory since $x \in y$ and $z \in x$ and y is well-founded (Fact 47).

The second claim follows from the first claim with extensionality. ∎

Fact 50 If x is well-founded, then $\sigma x \notin x$ and $\sigma x \neq x$.

Proof Follows from the absence of loops since $x \in \sigma x$. ∎

Exercise 51 Let $x \in y$ and x be well-founded. Prove $y \notin \sigma x$.

Exercise 52 Show that no HF set agrees with HF.

Exercise 53 Show that HF is large if every set is an HF set.

Exercise 54 (Minimal Element) Let p be an inhabited class of well-founded sets. Show that there is some x in p such that x contains no element of p. Use excluded middle.

9 HFT Characterisation of Numerals

An **HFT set** is a transitive HF set whose elements are transitive:

$$\text{HFT}\, x \;:=\; \text{HF}\, x \wedge \text{trans}\, x \wedge x \subseteq \text{trans}$$

Clearly, every numeral is an HFT set. We now show that every HFT set is a numeral. The HFT characterisation makes it clear that numerals are intimately linked with the notion of transitive sets. We start with the key lemma.

Lemma 55 Let u be the greatest element of a transitive set x of numerals. Then $x = \sigma u$. Thus x is a numeral.

Proof We show $x = \sigma u$ with extensionality. The direction $\sigma u \subseteq x$ follows with the transitivity of x. For the other direction, assume $y \in x$. We show $y \in \sigma u$ by by trichotomy for y and u. The cases $y \in u$ and $y = u$ are obvious. The case $u \in y$ is contradictory since $y \subseteq u$ (u is greatest element of x) and thus $u \in u$. ■

Fact 56 Every transitive and finite set of numerals is a numeral.

Proof Let x be a transitive and finite set of numerals. By Facts 23 and 32 we know that x has a greatest element. Thus x is a numeral by Lemma 55. ■

Fact 57 **HFT** is closed.

Theorem 58 A set is an HFT set if and only if it is a numeral.

Proof The direction from right to left follows with Facts 17, 16, and 36. For the other direction let x be an HFT set. Then x is well-founded. We show by WF induction on x that x is a numeral. By the inductive hypothesis and Fact 57 we know that every element of x is a numeral. Thus x is a numeral by Fact 56. ■

Note the use of WF induction in the proof of the theorem. It seems that HF induction does not suffice for a proof of the theorem.

10 Union

We now assume an operation $\bigcup x$ called **union** that yields the least set containing all elements of the elements of x.

$$\bigcup \;:\; \text{set} \to \text{set}$$
$$\text{Union} \;:\; \forall xz.\, z \in \bigcup x \leftrightarrow \exists y \in x.\, z \in y$$

It will be convenient to use an upper bound characterisation of unions. A set u is an **upper bound** of a set x if every element of x is a subset of u. An upper bound u of x is a **least upper bound** (**lub**) of x if u is a subset of every upper bound of x. Because set inclusion is a partial order, a set has at most one least upper bound.

Fact 59 (Lub Characterisation) $\bigcup x$ is the least upper bound of x.

Fact 59 is very useful for proofs:

1. $u \subseteq \bigcup x$ if $u \in x$.
2. $\bigcup x \subseteq u$ if u is an upper bound of x.
3. $\bigcup x = u$ if u is a least upper bound of x.

Fact 60 (Greatest Elements)

1. If u is the greatest element of x, then $u = \bigcup x$.
2. If $\bigcup x \in x$, the $\bigcup x$ is the greatest element of x.

Fact 61 $\bigcup \emptyset = \emptyset$ and $\bigcup \{x\} = x$.

Fact 62 $\bigcup x \subseteq x$ iff x is transitive.

Fact 63 (Predecessor) $\bigcup(\sigma x) = x$ if x is transitive.

Fact 64 $\bigcup x$ is transitive if every element of x is transitive.

Fact 65 $\bigcup x$ is well-founded if x is well-founded.

Fact 66 (Adjunction)

1. $\bigcup(\emptyset.x) = \bigcup x$ and $\bigcup((a.x).y) = a.\bigcup(x.y)$.
2. $\bigcup(x.y) = \bigcup y$ if $x \subseteq \bigcup y$.
3. $\bigcup(x.y) = \bigcup x$ if $\bigcup y \subseteq x$.

Fact 67 $\bigcup x$ is HF if x is HF.

Proof By HF induction on x. The case $x = \emptyset$ follows with Fact 61. Let $x = y.x$. We show $\bigcup(y.x)$ is HF by nested HF induction on y. The case $y = \emptyset$ follows with Fact 66 and the inductive hypothesis for x. Let $y = a.y$. The claim that $\bigcup((a.y).x)$ is HF follows with Fact 66 and the inductive hypothesis for y. ∎

Fact 68 (Monotonicity) If $x \subseteq y$, then $\bigcup x \subseteq \bigcup y$.

Exercise 69 Show that $\bigcup x$ is the greatest element of x if x is an inhabited finite chain.

11 Replacement, Separation, Description

An important axiom in set theory is replacement, which ensures the existence of functional images. With replacement we can define intersections of sets with classes and intersections of sets with sets.

We assume an operation $R@x$ called **replacement** that yields the **image** of a set x under a functional predicate R:

$$\_@\_ \; : \; (\text{set} \to \text{set} \to \mathbb{P}) \to \text{set} \to \text{set}$$
$$\text{Rep} \; : \; \text{functional } R \to \forall xz.\, z \in R@x \leftrightarrow \exists y \in x.\, Ryz$$

The restriction that the set $R@x$ is determined as image of x only if R is functional is essential. Dropping this restriction leads to inconsistency. The functionality restriction ensures that the image $R@x$ is at most as large as x. We may see the image $R@x$ as the set obtained from x be deleting all elements of x not in the domain of R and replacing all elements of x that are in the domain of R.

An important operation we can define with replacement is **separation**:

$$x \cap p \;:=\; (\lambda ab.pa \wedge a = b)@x$$

Fact 70 $z \in x \cap p \;\leftrightarrow\; z \in x \wedge pz.$

As the notation suggests, we may see $x \cap p$ as the intersection of x and p. In mathematics, the notation $\{\, a \in x \mid pa \,\}$ is used for a separation $x \cap p$. We define binary **intersection** of sets:

$$x \cap y \;:=\; \{\, z \in x \mid z \in y \,\}$$

With separation we can show that the notion of small and large classes are compatible with the inclusion ordering.

Fact 71 Every subclass of a small class is small. Hence every superclass of a large class is large.

Fact 72 (External Subset) Every set has a subset that is not an element of the set.

Proof Let x be set. Let $u := \{\, z \in x \mid z \notin z \,\}$. We have $u \subseteq x$. Suppose $u \in x$. Then $u \in u \leftrightarrow u \notin u$, which is contradictory. ∎

A class p is called **unique** if it has at most one element. Note that a binary predicate R on sets is functional iff the class Rx is unique for every set x.

An inhabited and unique class is called a **singleton class**. We define a **description operator** δp that yields the element of a singleton class:

$$\delta p \;:=\; \bigcup\, ((\lambda ab.pb)@\{\emptyset\})$$

Fact 73 Let p be a unique class containing x. Then $\delta p = x$.

With description we can define **projections** for ordered pairs.

$$\begin{aligned} \pi_1 u &:= \delta\,(\lambda x.\ \{x\} \in u) \\ \pi_2 u &:= \delta\,(\lambda y.\ u = \langle \pi_1 u, y\rangle) \end{aligned}$$

Fact 74 (Projections) $\pi_1\langle x, y\rangle = x$ and $\pi_2\langle x, y\rangle = y$.

For convenience, we define replacement for functions $f : \text{set} \to \text{set}$:

$$f@x \;:=\; (\lambda ab.fa = b)\,@\,x$$

Fact 75 $z \in f@x \;\leftrightarrow\; \exists a \in x.\ fa = z$.

In mathematics, the notation $\{\, fa \mid a \in x \,\}$ is used for $f@x$.
Finally, we define **cartesian products**:

$$x \times y \;:=\; \bigcup\,(\{\,\{\,\langle a, b\rangle \mid b \in y\,\} \mid a \in x\,\})$$

Fact 76 $z \in x \times y \;\leftrightarrow\; \exists a \in x\ \exists b \in y.\ z = \langle a, b\rangle$.

Exercise 77 (Large Classes) Prove the following:

1. The class of all sets is large.
2. The class of all finite sets is large.
3. Every subset-closed class is large.
   A class p is **subset-closed** if $\forall xy.\ px \to y \in x \to py$.

Exercise 78 Show that Num is small if HF is small.

Exercise 79 Show that $\delta(\lambda z.\ \text{agree}\ z\ \text{Num})$ agrees with Num if HF is small.

Exercise 80 (Functions as Sets) Given a function $f : \text{set} \to \text{set}$ and a set u, we can represent the restriction of f to u as the set $\{\,\langle x, fx\rangle \mid x \in u\,\}$. This fact makes it possible to develop set theory in a first-order logic not providing for functions. In fact, in Mathematics functions are typically defined as sets of pairs.

Define and verify two functions $\downarrow : (\text{set} \to \text{set}) \to \text{set} \to \text{set}$ and $\uparrow : \text{set} \to \text{set} \to \text{set}$ such that $(f \downarrow u) \uparrow x = fx$ for every $x \in u$.

Exercise 81 The replacement operator ensures that functional images of sets are sets. Assuming the existence of nonfunctional images would result in inconsistency. Give a predicate R and a set x such that the class $\lambda b.\exists a \in x.\ Rab$ is large.

Exercise 82 Together, replacement with functions, description, and separation can express general replacement. Assume a functional predicate R and prove

$$R@x = (\lambda a.\, \delta(Ra))\,@\,(x \cap (\lambda a.\, \exists b.\, Rab))$$

Exercise 83 Let x be a set and f be a function from sets to sets. Prove that there exists a set $u \subseteq x$ such that $u \notin f@x$. Hint: The claim to be shown is a variant of Cantor's Theorem and generalizes Fact 72 (take the identity function for f).

Exercise 84 (Subsets and XM) You may have noticed that proving that subsets of finite sets are finite requires excluded middle. With separation we can show that the assumption that subsets of finite sets are finite entails excluded middle.

Let $\tau P := \{\emptyset\} \cap (\lambda x.P)$. Note that τ is a function mapping propositions to sets. Prove the following:

a) τP is inhabited iff P holds.

b) Excluded middle holds for all propositions if every subset of $\{\emptyset\}$ is finite.

12 Diaconescu's Theorem

Diaconescu's Theorem [4] says that a set theory with a **choice function**

$$\gamma \;:\; \text{set} \to \text{set}$$
$$\text{Choice} \;:\; \forall x.\, \text{inhab}\, x \to \gamma x \in x$$

which for every inhabited set yields an element of the set entails excluded middle. Thus the standard set theory ZFC is inherently classical.

Theorem 85 (Diaconescu)
The presence of a choice function entails excluded middle.

Proof Let P be a proposition. We define $fx := \{\, z \in \sigma(\sigma\emptyset) \mid z = x \vee P \,\}$. Let γ be a choice function. Then $\gamma(f\emptyset) \in f\emptyset$ and $\gamma(f(\sigma\emptyset)) \in f(\sigma\emptyset)$. By the definition of f we have either P or $\gamma(f\emptyset) = \emptyset$ and $\gamma(f(\sigma\emptyset)) = \sigma\emptyset$. In the first case we are done. In the second case we show $\neg P$. Assume P. Then $f\emptyset = f(\sigma\emptyset)$ by extensionality. Thus $\emptyset = \sigma\emptyset$. Contradiction. ■

Note that the proof uses all axioms introduced so far except for union.

13 Power

We now assume an operation $\mathcal{P}x$ that yields the **power set of** x:

$$\mathcal{P} \;:\; \text{set} \to \text{set}$$
$$\text{Power} \;:\; \forall xz.\, z \in \mathcal{P}x \leftrightarrow z \subseteq x$$

The axiom Power specifies $\mathcal{P}x$ as the set of all subsets of x.

Fact 86

1. $x \in \mathcal{P}x$.
2. $\mathcal{P}$ preserves transitivity: $\mathcal{P}x$ transitive if x is transitive.
3. $\mathcal{P}$ preserves well-foundedness: $\mathcal{P}x$ well-founded if x is well-founded.
4. $\mathcal{P}$ is monotone: $\mathcal{P}x \subseteq \mathcal{P}y$ if $x \subseteq y$.
5. Union undoes power: $\bigcup(\mathcal{P}x) = x$.
6. $\mathcal{P}$ is injective: $x = y$ if $\mathcal{P}x = \mathcal{P}y$.
7. x is transitive if and only if $x \subseteq \mathcal{P}x$.
8. $\mathcal{P}x = x \cup \mathcal{P}x$ if x is transitive.

Exercise 87 Prove $\mathcal{P}x \notin x$.

14 Subsets of Finite Sets

We now show that every subset of a finite set is a finite set, and that the power set of an HF set is an HF set. Both results require excluded middle. So far we have not used excluded middle in our development.

The use of excluded middle can be packaged into a disjunctive result for subsets of adjunctions.

Lemma 88 Let $x \subseteq a.y$. Then either $x \subseteq y$ or $x = a.x'$ for some $x' \subseteq y$.

Proof Case analysis using **excluded middle**.

Let $a \in y$. Then $x \subseteq y$.

Let $a \notin y$ and $a \in x$. Then $x = a.\{\, z \in x \mid z \neq a \,\}$.

Let $a \notin y$ and $a \notin x$. Then $x \subseteq y$. ∎

Fact 89 Every subset of a finite set is a finite set.

Proof Let x be a finite set. We prove by Fin induction on x that every finite subset of x is finite.

Let $x = \emptyset$. Then $\emptyset$ is the only subset of x and the claim follows.

Let $x = a.x'$ and $y \subseteq a.x'$. Case analysis by Lemma 88. If $y \subseteq x'$, the claim follows with the inductive hypothesis. Otherwise, let $y = a.x''$ for some $x'' \subseteq x'$. By the inductive hypothesis we know that x'' is finite. Hence y is finite. ∎

Next we show that HF sets are closed under taking power sets. The proof is by HF induction and verifies a recursive function computing power sets. For this, two simpler recursive functions for HF sets are needed, one computing binary unions and one computing replacements. There are similar functions for lists known as *append* and *map*. The verification of the power set function requires excluded middle as packaged by Lemma 88.

Lemma 90

1. $\emptyset \cup y = y$ and $(a.x) \cup y = a.(x \cup y)$.
2. $f@\emptyset = \emptyset$ and $f@(a.x) = fa.(f@x)$.
3. $\mathcal{P}\emptyset = \{\emptyset\}$ and $\mathcal{P}(a.x) = \mathcal{P}x \cup ((\lambda z.(a.z))@(\mathcal{P}x))$.

Proof All equations are verified using extensionality. The proofs are routine except for the direction from left to right of the adjoin equation for power, which assumes $y \subseteq a.x$. Using Lemma 88, the proof considers two easy cases. ∎

Fact 91

1. HF is closed under binary union.
2. HF is closed under replacement with $\lambda z.(a.z)$.
3. HF is closed under power.

Proof Each claim follows by HF induction using the appropriate equations from Lemma 90. Because of the second equation closure under power requires closure under binary union and replacement. ∎

15 Ordinals

The idea that a number is the set of all smaller numbers allows for more numbers than just the numerals. Suppose ω is a set agreeing with the class Num of numerals.[2] By extensionality we know that ω is the only such set. We may accept ω as a number given that it is the set of all smaller numbers (membership serves as smaller relation). Once we have accepted ω as a number, iterating the successor function on ω provides us with further numbers:

$$\emptyset \in \sigma\emptyset \in \sigma^2\emptyset \in \cdots \in \omega \in \sigma\omega \in \sigma^2\omega \in \cdots$$

One often refers to ω and its successors as *transfinite numbers*, and to $\emptyset$ and its successors as finite numbers.

We formalize the idea of transfinite numbers with an inductive class $\mathcal{O}$ whose members we call **ordinals**:

$$\frac{\mathcal{O}\,x}{\mathcal{O}\,(\sigma x)} \qquad\qquad \frac{x \subseteq \mathcal{O}}{\mathcal{O}\,(\bigcup x)}$$

[2]ZFC ensures the existence of ω.

Since $\bigcup\emptyset = \emptyset$, we know by the union rule that $\emptyset$ is an ordinal. With the successor rule we can obtain all further numerals.

Fact 92 Every numeral is an ordinal.

Fact 93 (Closedness) Every element of an ordinal is an ordinal.

Proof By ordinal induction. ■

Fact 94 Every ordinal is transitive and well-founded.

Proof By ordinal induction. Both union and successor preserve transitivity and well-foundedness (Facts 64, 17, 65, and 46). ■

Since σ is injective on well-founded sets (Fact 49), σ is injective on ordinals.

Fact 95 (Strictness) If x is an ordinal, then $x \notin x$.

Fact 96 (Largeness) If $x \subseteq \mathcal{O}$, then $\sigma(\bigcup x) \notin x$. Thus $\mathcal{O}$ is large.

Proof Let $x \subseteq \mathcal{O}$ and $\sigma(\bigcup x) \in x$. Then $\bigcup x \in \bigcup x$. Contradiction since $\bigcup x$ is an ordinal and ordinals are well-founded. ■

Fact 97 Suppose ω agrees with Num. Then:

1. ω is not a numeral.
2. $\bigcup\omega = \omega$
3. ω is an ordinal.

Closing ordinals under union ensures that ω (if it exists) is an ordinal and that the class of ordinals is maximal (in contrast to the class of numerals, which can be extended to the class of ordinals). We still have to show that the closure under union doesn't introduce unwanted ordinals. We start by showing that the class of ordinals is successor linear (i.e., either $x \subseteq y$ or $\sigma y \subseteq x$ for all ordinals x and y).

We prove successor linearity with the so-called double induction principle. This yields an elegant proof that generalizes to related problems. We establish the double induction principle with an inductive predicate:

$$\frac{\mathcal{O}x \qquad \mathcal{D}xy \qquad \mathcal{D}yx}{\mathcal{D}(\sigma x)y} \qquad\qquad \frac{\forall z.\, z \in x \to \mathcal{D}zy}{\mathcal{D}(\bigcup x)y}$$

Lemma 98 (Double Induction) $\mathcal{D}xy$ holds for all ordinals x and y.

Proof Let x be an ordinal. We proof $\forall y.\, \mathcal{O}y \to \mathcal{D}xy$ by ordinal induction on x.

1. Let $\mathcal{O}x$ and $\mathcal{O}y$. We prove $\mathcal{D}(\sigma x)y$. By unfolding of $\mathcal{D}$ it suffices to prove $\mathcal{D}xy$ and $\mathcal{D}yx$. $\mathcal{D}xy$ follows with the inductive hypothesis for x. We show $\mathcal{D}yx$ by ordinal induction on y.
   a) Let $\mathcal{O}y$. We prove $\mathcal{D}(\sigma y)x$. By unfolding of $\mathcal{D}$ it suffices to prove $\mathcal{D}yx$ and $\mathcal{D}xy$, which follow by the inductive hypothesis for y and the inductive hypothesis for x.
   b) Let $y \subseteq \mathcal{O}$. We prove $\mathcal{D}(\bigcup y)x$. By unfolding of $\mathcal{D}$ it suffices to prove $\mathcal{D}zx$ for all $z \in y$, which holds by the inductive hypothesis for y.
2. Let $x \subseteq \mathcal{O}$. We prove $\mathcal{D}(\bigcup x)y$. By unfolding of $\mathcal{D}$ it suffices to prove $\mathcal{D}zy$ for all $z \in x$, which holds by the inductive hypothesis for x. ■

We can now prove successor linearity if we assume excluded middle. A proof not using excluded middle is not known.

Theorem 99 (Successor Linearity)
Let x and y be ordinals. Then either $x \subseteq y$ or $\sigma y \subseteq x$.

Proof By Lemma 98 we have $\mathcal{D}xy$. We prove the claim by induction on $\mathcal{D}xy$.

1. Let x be an ordinal, $x \subseteq y \vee \sigma y \subseteq x$, and $y \subseteq x \vee \sigma x \subseteq y$. We show $\sigma x \subseteq y \vee \sigma y \subseteq \sigma x$. All four cases are obvious.
2. Let $z \subseteq y \vee \sigma y \subseteq z$ for all $z \in x$. We show $\bigcup x \subseteq y \vee \sigma y \subseteq \bigcup x$. By **excluded middle** we have two cases.
   a) y is an upper bound of x. Then $\bigcup x \subseteq y$.
   b) There is $z \in x$ such that $z \not\subseteq y$. We show $\sigma y \subseteq \bigcup x$. By assumption we have either $z \subseteq y$ or $\sigma y \subseteq z$. The first case is contradictory since $z \not\subseteq y$. Let $\sigma y \subseteq z$. It suffices to show $z \subseteq \bigcup x$, which holds since $z \in x$. ■

Corollary 100 (Epsilon Linearity)
Let x and y be ordinals. Then either $x \subseteq y$ or $y \in x$.

Proof By successor linearity we have either $x \subseteq y$ or $\sigma y \subseteq x$. The claim follows since $y \in \sigma y$. ■

Corollary 101 (Linearity) Let x and y be ordinals. Then either $x \subseteq y$ or $y \subseteq x$.

Proof Follows from epsilon linearity since x is transitive. ■

Fact 102 (Trichotomy)
Let x and y be ordinals. Then either $x \in y$ or $x = y$ or $y \in x$.

Proof Follows with strict linearity (twice) and extensionality. ■

Fact 103 (Cumulativity) Let x and y be ordinals. Then $x \in y \leftrightarrow x \subseteq y \wedge x \neq y$.

Proof The direction from left to right follows with transitivity and strictness. The other direction follows with strict linearity and extensionality. ■

Fact 104 (Monotonicity) Let x and y be ordinals. Then:

1. If $x \subseteq y$, then $\sigma x \subseteq \sigma y$.
2. If $x \in y$, then $\sigma x \in \sigma y$.

Proof Let $x \subseteq y$. We show $\sigma x \subseteq \sigma y$. By successor linearity we have either $y \subseteq x$ or $\sigma x \subseteq y$. If $y \subseteq x$, we have $x = y$ by extensionality and the claim is trivial. If $\sigma x \subseteq y$, the claim follows with $y \subseteq \sigma y$.

Let $x \in y$. We show $\sigma x \in \sigma y$. Strict linearity yields either the claim or the assumption $\sigma y \subseteq \sigma x$. By $x \in y$ and transitivity of y we have $x \subseteq y$. Thus $\sigma x \subseteq \sigma y$ by (1) and $\sigma x = \sigma y$ by extensionality. Thus $x = y$ by injectivity of σ. Thus $x \in x$ contradicting strictness. ■

We distinguish between successor and limit ordinals. A **successor ordinal** is an ordinal that can be obtained as successor of some ordinal, and a **limit ordinal** is an ordinal that is a fixed point of union. It turns out that every ordinal is either a successor or a limit ordinal, but not both.

Fact 105 (Successor-Limit Distinction)
Let x be an ordinal. Then either $x = \sigma(\bigcup x)$ or $x = \bigcup x$.

Proof By closedness we know that $\bigcup x$ is an ordinal. Thus $\sigma(\bigcup x)$ is an ordinal. We prove the claim by case analysis using trichotomy for x and $\sigma(\bigcup x)$.

Let $x \in \sigma(\bigcup x)$. Then $x = \bigcup x$ or $x \in \bigcup x$. Let $x \in \bigcup x$. We have $\bigcup x \subseteq x$ since x is transitive. Thus $x \in x$ contradicting strictness.

Let $x = \sigma(\bigcup x)$. The claim follows.

Let $\sigma(\bigcup x) \in x$. Then $\sigma(\bigcup x) \subseteq \bigcup x$. Thus $\bigcup x \in \bigcup x$ contradicting strictness. ■

Fact 106 (Least Ordinals) Every class containing an ordinal contains a least ordinal.

Proof Let p be a class and x be an ordinal satisfying p. We prove by WF induction on x that there is an ordinal u satisfying p such that $u \subseteq z$ for every ordinal z satisfying p. Case analysis using **excluded middle** on $\exists y \in x.\ \mathcal{O}y \wedge py$. In the positive case, the inductive hypothesis yields the claim. In the negative case, we show that x is the least ordinal satisfying p. Let y be a ordinal satisfying p. We show $x \subseteq y$. Case analysis by epsilon linearity. The nontrivial case $y \in x$ contradicts the outer assumption $\neg\exists y \in x.\ py$. ■

The same proof works for numerals. One can show that the existence of least numerals entails excluded middle.

Note that there were only two direct uses of excluded middle in this section (proof of successor linearity, Theorem 99, and existence of least ordinals, Fact 106).

Exercise 107 Assume excluded middle and show that either u is an upper bound of x or there is some $y \in x$ such that $y \notin u$.

Exercise 108 Let ω be the set of all numerals. Show that ω is the least limit ordinal.

16 WFT Characterisation of Ordinals

Fact 109 Every transitive set of ordinals is an ordinal.

Proof Let x be a transitive set of ordinals. Let y be the least ordinal such that $y \notin x$ (exists by Facts 96 and 106). We show $x = y$ by extensionality.

Let $z \in x$. We show $z \in y$. Case analysis by trichotomy for z in y. For the two nontrivial cases we show $y \in x$, which contradicts $y \notin x$. If $z = y$, $y \in x$ is immediate. If $y \in z$, we have $y \in x$ since x is transitive.

Let $z \in y$. We show $z \in x$ by contradiction using **excluded middle**. Let $z \notin x$. Since z is an ordinal by closedness, we have $y \subseteq z$ since y is the least ordinal that is not in x. Thus $z \in z$ contradicting strictness. ∎

Theorem 110 A set is an ordinal if and only if it is well-founded, transitive, and all its elements are transitive.

Proof The direction from left to right follows with closedness since every ordinal is transitive and well-founded. For the other direction, let x be a transitive and well-founded set such that every element of x is transitive. We show by WF induction on x that x is an ordinal. By Fact 109 it suffices to show that x is a transitive set of ordinals. Since x is transitive by assumption, it remains to show that x is a set of ordinals. This follows with the inductive hypothesis since every element of x is a transitive set of transitive sets. ∎

Exercise 111 A *WFT set* is a transitive and well-founded set containing only transitive elements. Show that every element of a WFT set is a WFT set.

17 Finite Ordinals

We know that every numeral is a finite ordinal. We now show that every finite ordinal is a numeral.

Fact 112 (Infinity) Every set containing all numerals is infinite.

Proof Let x be a set containing all numerals. Suppose x is finite. Then $x \cap \text{Num}$ is an inhabited finite chain (Facts 89 and 23). Let a be the greatest element of $x \cap \text{Num}$ (exists by Fact 32). Then σa is a numeral and thus $\sigma a \in x$. Hence $\sigma a \subseteq a$ and thus $a \in a$. Contradiction by Fact 19. ∎

Fact 113 (Tightness) An ordinal is a numeral if it is a subset of a numeral.

Proof Let x be an ordinal and y be a numeral such that $x \subseteq y$. We prove by numeral induction on y that x is a numeral.

If $y = \emptyset$, then $x = \emptyset$ and thus x is a numeral.

Let $y = \sigma y'$. Case analysis using successor linearity for x and y'. If $x \subseteq y'$, the claim follows by the inductive hypothesis. If $\sigma y' \subseteq x$, x is a numeral since $x = y$ by extensionality. ∎

Fact 114 (Finite Ordinals) Every finite ordinal is a numeral.

Proof Let x be a finite ordinal. Then x does not contain all numerals (Fact 112). By excluded middle there exists a numeral $a \notin x$. We have $x \subseteq a$ by epsilon linearity for ordinals (Corollary 100). Hence x is a numeral by Fact 113. ∎

18 Cumulative Hierarchy

We define an inductive class $\mathcal{S}$ whose elements we call **stages**:

$$\frac{\mathcal{S}\,x}{\mathcal{S}\,(\mathcal{P}x)} \qquad\qquad \frac{x \subseteq \mathcal{S}}{\mathcal{S}\,(\bigcup x)}$$

The definition of stages mimics the definition of ordinals by replacing the successor function with the power function. It turns out that stages enjoy the same order properties as ordinals. The difference between ordinals and stages is that σx only adds x to x, while $\mathcal{P}x$ adds all subsets of x to x (provided x is transitive). It will turn out that every well-founded set appears as element of some stage. This means that power and union generate exactly the well-founded sets.

The class of stages is known as **cumulative hierarchy**.

Fact 115 Every stage is transitive and well-founded. Thus $x \notin x$ if x is a stage.

Fact 116 (Largeness) If $x \subseteq \mathcal{S}$, then $\mathcal{P}(\bigcup x) \notin x$. Thus $\mathcal{S}$ is large.

Proof Let $x \subseteq \mathcal{S}$ and $\mathcal{P}(\bigcup x) \in x$. Then $\bigcup x \in x$ and thus $\bigcup x \in \bigcup x$. Contradiction since $\bigcup x$ is a stage and stages are well-founded. ∎

Theorem 117 (Successor Linearity)
Let x and y be stages. Then either $x \subseteq y$ or $\mathcal{P}y \subseteq x$.

Proof The proof is identical to the proof of successor linearity for ordinals (Theorem 99). One first proves a double induction lemma. ∎

We call a set **reachable** if it is the element of a stage. We wil show that every well-founded set is reachable. To do so, we define the **rank predicate**:

$$\rho a x \;:=\; \mathcal{S}x \wedge a \subseteq x \wedge a \notin x$$

We will show that the rank predicate is functional on all sets and total on well-founded sets. Totality of ρ implies that all well-founded sets are reachable. Functionality of ρ justifies that we speak of the **rank** of a set.

Fact 118 Let $\rho a x$. Then x is the least stage such that $a \subseteq x$.

Proof Let y be a stage such that $a \subseteq y$ and $a \notin y$. We show $x \subseteq y$ by successor linearity for the stages x and y. Let $\mathcal{P}y \subseteq x$. Since $a \subseteq y$, we have $a \in x$, contradicting the assumption $\rho a x$. ∎

Fact 119 ρ is functional.

Lemma 120 Every reachable set has a rank.

Proof Let x be a stage and $a \in x$. We show by stage induction on x that a has a rank.

Let $x = \mathcal{P}y$ for some stage y. Then $a \subseteq y$ since $a \in x$. Case analysis on $a \in y$ using **excluded middle**. In the positive case, the claim follows with the inductive hypothesis. In the negative case, we have $\rho a y$.

Let $x = \bigcup u$ for some set u of stages. Then $a \in y \in u$ for some stage y. The claim follows with the inductive hypothesis. ∎

Lemma 121 Every set of reachable sets is reachable.

Proof Let every element of x be reachable. We show $x \in \mathcal{P}(\mathcal{P}(\bigcup(\rho @ x)))$. This yields the claim since $\rho @ x$ is a set of stages since ρ is functional. It suffices to show $x \subseteq \mathcal{P}(\bigcup(\rho @ x))$. Let $y \in x$. We show $y \subseteq \bigcup(\rho @ x)$. We have $\rho y a$ for some a since x contains only reachable sets and every reachable set has a rank (Lemma 120). The claim follows since $y \subseteq a$ and $a \in \rho @ x$. ∎

Lemma 122 Every well-founded set is reachable.

Proof Let x be a well-founded set. We prove by WF induction on x that x is reachable. The claim follows by Lemma 121 and the inductive hypothesis. ∎

Theorem 123 Let x be a set. The following properties are equivalent:

1. x is well-founded.
2. x is an element of a stage.
3. x has a rank.

Proof Follows with Lemmas 122 and 120 and Fact 115 ∎

Exercise 124 (Least Stages) Let x be a stage and p be a class such that px. Show that there exists a stage y in p such that $y \subseteq z$ for every stage z in p. You may use excluded middle.

Exercise 125 (Successor-Limit Distinction)
Let x be a stage. Prove that either $x = \sigma(\bigcup(x \cap \mathcal{S}))$ or $x = \bigcup(x \cap \mathcal{S})$.

19 Finite Cumulative Stages

We identify the **basic stages** of the cumulative hierarchy with an inductive predicate BS:

$$\frac{}{\mathrm{BS}\,\emptyset} \qquad\qquad \frac{\mathrm{BS}\,x}{\mathrm{BS}\,(\mathcal{P}x)}$$

We will show that the basic stages are exactly the finite stages. We will also show that the HF sets are exactly the sets appearing as elements of the basic stages.

Fact 126 (BS $\subseteq$ $\mathcal{S}$) Every basic stage is a stage.
Thus every basic stage is transitive and well-founded.

Fact 127 (BS $\subseteq$ HF) Every basic stage is an HF set.

Proof Follows with Fact 91. ∎

Fact 128 (Infinity) Every set containing all basic stages is infinite.

Proof Analogous to Fact 112. ∎

Fact 129 (Infinity) Every set containing all HF sets is infinite.

Proof Follows with Facts 128 and 127. ∎

Fact 130 (Tightness) Every stage that is a subset of a basic stage is basic.

Proof Analogous to Fact 113. ∎

Fact 131 (Finite Stages) Every finite stage is basic.

Proof Analogous to Fact 114. ∎

Theorem 132 A stage is finite if and only if it is basic.

Proof Follows with Facts 131 and 127. ∎

We now know that the finite stages are exactly the basic stages. We also know that every basic stage is an HF set. It remains to show that every HF set appears as an element of a basic stage.

Fact 133 Every HF set is an element of a basic stage.

Proof Let x be an HF set. We prove by HF induction on x that there exists a basic stage y such that $x \in y$.

Let $x = \emptyset$. Then x is an element of $\mathcal{P}\emptyset$.

Let $x = a.y$. By the inductive hypothesis we have basic stages u and v such that $a \in u$ and $y \in v$. Case analysis by stage linearity for u and v.

Let $u \subseteq v$. Then $a.y \in \mathcal{P}v$ since v is transitive.

Let $v \subseteq u$. Then $a.y \in \mathcal{P}u$ since u is transitive. ∎

Theorem 134 A set is in HF if and only if it is an element of some basic stage.

Proof Follows with Facts 133 and 127. ∎

Corollary 135 Let x agree with BS. Then $\bigcup x$ is a stage agreeing HF.

20 Towers

We generalise the definition of the inductive classes for ordinals and stages using some function g : set → set:

$$\frac{\mathcal{T}\,x}{\mathcal{T}\,(gx)} \qquad\qquad \frac{x \subseteq \mathcal{T}}{\mathcal{T}\,(\bigcup x)}$$

We call the inductive class $\mathcal{T}$ the **tower for g** and the elements of $\mathcal{T}$ the **stages for g**. We refer to g as the **generator** of the tower. If we take σ as generator, we get the class of ordinals, and if we take $\mathcal{P}$ as generator, we get the stages of the cumulative hierarchy. If the generator satisfies certain properties, the stages of the corresponding tower will satisfy the properties we have seen for ordinals and cumulative stages.

Fact 136 If g preserves well-foundedness, then every stage of $\mathcal{T}$ is well-founded.

Fact 137 If g preserves transitivity, then every stage of $\mathcal{T}$ is transitive.

We call g **increasing on $\mathcal{T}$** if $x \subseteq gx$ for all stages x of $\mathcal{T}$.

Fact 138 (Successor Linearity) Let g be increasing on $\mathcal{T}$ and let x and y be stages of $\mathcal{T}$. Then either $x \subseteq y$ or $gy \subseteq x$.

Corollary 139 (Linearity) Let g be increasing on $\mathcal{T}$ and let x and y be stages of $\mathcal{T}$. Then either $x \subseteq y$ or $y \subseteq x$.

We call g **eager** if $x \in gx$ for every set x.

Fact 140 $\mathcal{T}$ is large if g is eager and preserves well-foundedness.

Fact 141 If g is eager and preserves transitivity, then g is increasing on $\mathcal{T}$.

Fact 142 If g is eager and preserves transitivity and well-foundedness, and x and y are stages, then $x \in y$ if and only if $x \subset y$.

We define an inductive class $\mathcal{T}_0$ of **basic stages**:

$$\frac{}{\mathcal{T}_0\,\emptyset} \qquad\qquad \frac{\mathcal{T}_0\,x}{\mathcal{T}_0\,(gx)}$$

We have $\mathcal{T}_0 \subseteq \mathcal{T}$.

Fact 143 Let g be increasing on $\mathcal{T}$. Then every stage that is a subset of a basic stage is basic.

Fact 144 Let g be eager, increasing on $\mathcal{T}$, and preserve well-foundedness. Then there are infinitely many basic stages and every finite stage is basic.

The towers for ordinals and cumulative stages have the least element property (Fact 106 for ordinals and Exercise 124 for cumulative stages). In both cases the proof is by WF induction exploiting that ordinals and cumulative stages are well-founded sets. It turns out that least elements exist for all towers obtained with increasing generators. In the general setting we do not know whether the stages are well-founded and hence cannot use WF induction. However, we can establish an induction principle similar to WF induction for the stages of every tower obtained with an increasing generator. We will need the general result for a tower we will use for the proof of the well-ordering theorem (Theorem 163).

We define an inductive class $\mathcal{A}$ of **accessible sets**:

$$\frac{\forall y.\ \mathcal{T}\,y \to y \subset x \to \mathcal{A}\,y}{\mathcal{A}\,x}$$

Informally, a set x is accessible if there exists no infinite descending chain of stages issuing from x. For accessible sets we have an induction principle called **complete induction** that is similar to WF induction.[3] Complete induction says that when we prove that an accessible set x satisfies a property p, we can assume (as inductive hypothesis) that every stage that is a proper subset of x satisfies p.

Fact 145 (Complete Induction) Let g be increasing on $\mathcal{T}$. Then $\mathcal{T} \subseteq \mathcal{A}$.

Proof We show $\forall x.\ \mathcal{T}x \to \forall y.\ y \subseteq x \to \mathcal{A}y$ by induction on $\mathcal{T}x$. It is essential to show this generalized claim.

Let $\mathcal{T}x$ and $y \subset gx$. We show $\mathcal{A}y$. Unfolding the definition of $\mathcal{A}$, we assume $\mathcal{T}z$ and $z \subset y$ and show $\mathcal{A}z$. By successor linearity we have $z \subseteq x$. Hence $\mathcal{A}z$ by the inductive hypothesis for x.

Let $x \subseteq \mathcal{T}$ and $y \subseteq \bigcup x$. We show $\mathcal{A}y$. Unfolding the definition of $\mathcal{A}$, we assume $\mathcal{T}z$ and $z \subset y$ and show $\mathcal{A}z$. We observe that z is not an upper bound of x (if it was, we have $y \subset \bigcup x \subseteq z \subset y$). Hence $a \nsubseteq z$ for some $a \in x$ by **excluded middle**. Thus $z \subset a$ by successor linearity. Hence $\mathcal{A}z$ by the inductive hypothesis for $a \in x$. ∎

Fact 146 (Least Stages) Let g be increasing on $\mathcal{T}$. Then every class containing a stage of $\mathcal{T}$ contains a least such stage.

Proof Let px and $\mathcal{T}x$. We show by complete induction on x (using Fact 145) that there is a least y such that py and $\mathcal{T}y$. If x has this property we are done. Otherwise, by **excluded middle**, we have an x' such that px', $\mathcal{T}x'$, and $x \nsubseteq x'$. By successor linearity we have $x' \subset x$. Now the claim follows by the inductive hypothesis for x'. ∎

Exercise 147 Prove that g is injective on $\mathcal{T}$ if g is eager, increasing on $\mathcal{T}$, and preserves well-foundedness.

Exercise 148 (Constructive Proofs for Basic Stages) Show successor linearity and accessibility for basic stages not using excluded middle. Hint: Define a suitable double induction predicate for basic stages.

[3] We speak of complete induction because there is a similarity with complete induction for natural numbers.

21 Closures and Infinity Axiom

With the assumptions made so far, we cannot prove that an infinite set exists. We change this situation by assuming the **infinity axiom**:

$$\text{Inf} \;:\; \exists x.\, 0 \in x \wedge \forall z.\, z \in x \to \sigma z \in x$$

The infinity axiom asserts the existence of a set containing all numerals. By Fact 112 we know that such a set is infinite.

Let o be a set and f be a function mapping sets to sets. We say that a set x is **closed under o and f** if $o \in x$ and $fy \in x$ whenever $y \in x$. Note that the infinity axiom asserts the existence of a set closed under $\emptyset$ and σ.

We say that a set is the **closure of o and f** if it is the least set closed under o and f. We will show that the closure of o and f always exists. We will also show that the closure of $\emptyset$ and σ is the set of all numerals, and that the closure of $\emptyset$ and $\mathcal{P}$ is the set of all basic stages. By Theorem 134 we then know that the union of the closure of $\emptyset$ and $\mathcal{P}$ is the set of all HF sets.

Lemma 149 (Conditional Existence)
The closure of o and f exists if a set closed under o and f exists.

Proof Let u be closed under o and f. With the separation we obtain the subset $v \subseteq u$ containing all elements of u that are in every set closed under o and f. It is easy to see that v is the least set closed under o and f. ■

Fact 150 (Closure Induction) Let x be the closure of o and f and p be a class. Then $x \subseteq p$ if po and $p(fx)$ whenever px.

Proof Let po and $p(fx)$ whenever px. Then $x \cap p$ is closed under o and f. Thus $x \subseteq x \cap p$ since x is the least such set. Hence $x \subseteq p$. ■

We define a **closure operator** $C : \text{set} \to (\text{set} \to \text{set}) \to \text{set}$ as follows:

$$C[o,f] \;:=\; \delta\,(\lambda x.\ x \text{ is the closure of } o \text{ and } f)$$

Lemma 151 $C[o,f]$ is the closure of o and f if a set closed under o and f exists.

Proof Follows with Fact 73, Lemma 149, and the uniqueness of closed sets. ■

Fact 152 $C[\emptyset,\sigma]$ is the closure of $\emptyset$ and σ.

Proof Follows with Lemma 151 and the infinity axiom. ■

We can now define the set

$$\omega := C[\emptyset, \sigma]$$

containing exactly the numerals.

Fact 153 (Omega) ω contains exactly the numerals.

Proof By Fact 152 we know that $C[\emptyset, \sigma]$ is the closure of $\emptyset$ and σ. By numeral induction it follows that every numeral is in ω, and by closure induction (Fact 150) it follows that every element of ω is a numeral. ∎

We still have to show that all closures exist. Given o and f, we will define a functional predicate R_{of} mapping the numerals to the sets o, fo, $f(fo)), \dots$. The image $R_{of}@\omega$ will then give us the closure of o and f.

Given o and f, we define an inductive predicate R_{of}:

$$\frac{}{R_{of}\ \emptyset\ o} \qquad\qquad \frac{R_{of}\ x\ y}{R_{of}\ (\sigma x)\ (fy)}$$

Lemma 154 If $R_{of}\ x\ y$, then x is a numeral.

Proof By R_{of} induction. ∎

Lemma 155 R_{of} is functional.

Proof We write R for R_{of}. Let Rxy. We prove by R_{of} induction that Rxz implies $z = y$. The base case is straightforward. In the successor case we have $x = \sigma x'$, $y = fy'$, $Rx'y'$, and $R(\sigma x')z$. We show $z = fy'$. By inversion of $R(\sigma x')z$ we obtain x'' and z' such that $\sigma x' = \sigma x''$, $z = fz'$ and $Rx''z'$. By the inductive hypothesis it suffices to show $x' = x''$, which follows by injectivity of σ on numerals and Lemma 154. ∎

Lemma 156 $R_{of}@\omega$ is closed under o and f.

Proof Straightforward consequence of Lemma 155. ∎

Theorem 157 $C[o, f]$ is the closure of o and f.

Proof Follows with Lemmas 151 and 156. ∎

Fact 158 $C[\emptyset, \mathcal{P}]$ is the set of all basic stages.

Proof Follows with Theorem 157, closure induction, and BS induction. ∎

Fact 159 $\bigcup(C[\emptyset, \mathcal{P}])$ is the set of all HF sets.

Proof Follows with Theorem 134 and Fact 158. ∎

Fact 160 (Transitive Closure)
Let x be a set. Then $\mathrm{tc}\, x := \bigcup(C[x, \cup])$ is the least transitive superset of x.

Proof We have $x \subseteq \mathrm{tc}\, x$ since $x \in C[x, \cup]$.

We show that $\mathrm{tc}\, x$ is transitive. Let $z \in y \in C[x, \cup]$. We show $z \subseteq \mathrm{tc}\, x$. It suffices to show $\bigcup y \in C[x, \cup]$, which holds since $y \in C[x, \cup]$.

Let $x \subseteq y$ and y be transitive. We show $\mathrm{tc}\, x \subseteq y$. It suffices to show that $\forall z \in C[x, \cup].\ z \subseteq y$. We show this by closure induction. For $z = x$, we have $x \subseteq y$ by assumption. For $z = \bigcup z'$, we have $z' \subseteq y$ by the inductive hypothesis and need to show $\bigcup z' \subseteq y$. Holds since y is an upper bound for z' since y is transitive. ∎

Exercise 161 Prove that a set is well-founded if and only if its transitive closure is well-founded.

Exercise 162 Prove that the classes Num, HF, and BS are all small if one of them is small. Do the proof without using the infinity axiom and its consequences.

22 Review

We review the axioms and the most important definitions of the set theory presented so far.

Constants

set	:	Type	
$\_ \in \_$	:	$\mathrm{set} \to \mathrm{set} \to \mathbb{P}$	membership
$\emptyset$	:	set	empty set
$\_ \cdot \_$	:	$\mathrm{set} \to \mathrm{set} \to \mathrm{set}$	adjunction
$\bigcup$	:	$\mathrm{set} \to \mathrm{set}$	union
$\_ @ \_$	:	$(\mathrm{set} \to \mathrm{set} \to \mathbb{P}) \to \mathrm{set} \to \mathrm{set}$	replacement
$\mathcal{P}$	:	$\mathrm{set} \to \mathrm{set}$	power

Variables

x, y, z, a, b, c, o	:	set	set
p, q	:	$\mathrm{set} \to \mathbb{P}$	class
R	:	$\mathrm{set} \to \mathrm{set} \to \mathbb{P}$	relation
f	:	$\mathrm{set} \to \mathrm{set}$	

Definitions

$x \subseteq y$	$:=$	$\forall z \in x.\ z \in y$	inclusion
$x \subseteq p$	$:=$	$\forall z \in x.\ pz$	inclusion
σx	$:=$	$x.x$	successor
$x \cap p$	$:=$	$(\lambda ab.\ a = b \wedge pa) @ x$	separation
$x \cap y$	$:=$	$x \cap (\lambda z.\ z \in y)$	intersection
$f@x$	$:=$	$(\lambda ab.\ fa = b) @ x$	replacement
δp	$:=$	$\bigcup ((\lambda a.p) @ \sigma\emptyset)$	description
transitive x	$:=$	$\forall y \in x.\ y \subseteq x$	
unique p	$:=$	$\forall xy.\ px \rightarrow py \rightarrow x = y$	
functional R	$:=$	$\forall x.$ unique (Rx)	
least p x	$:=$	$px \wedge \forall z.\ pz \rightarrow x \subseteq z$	
closed o f x	$:=$	$o \in x \wedge \forall z.\ z \in x \rightarrow fz \in x$	
$C[o, f]$	$:=$	δ (least (closed o f))	closure
ω	$:=$	$C[\emptyset, \sigma]$	
tc x	$:=$	$\bigcup (C[x, \cup])$	transitive closure

Axioms

$x \subseteq y \rightarrow y \subseteq x \rightarrow x = y$		extensionality
$z \notin \emptyset$		empty set
$z \in x.y \ \leftrightarrow\ z = x \vee z \in y$		adjunction
$z \in \bigcup x \ \leftrightarrow\ \exists y \in x.\ z \in y$		union
$z \in R@x \ \leftrightarrow\ \exists y \in x.\ Ryz$	if functional R	replacement
$z \in \mathcal{P}x \ \leftrightarrow\ z \subseteq x$		power
$\exists x.\ \emptyset \in x \wedge \forall z \in x.\ \sigma z \in x$		infinity

Notations and Ordered Pairs

$$\{x_1, \ldots, x_n\} := x_1. \cdots .x_n.\emptyset$$

$$\{\, z \in x \mid pz \,\} := z \cap p$$

$$\{\, fz \mid z \in x \,\} := f@x$$

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}$$

$$x \times y := \bigcup \{\, \{\, \langle a, b \rangle \mid b \in y \,\} \mid a \in x \,\}$$

Inductive Classes

$$\frac{}{\text{Fin}\,\emptyset} \qquad \frac{\text{Fin}\,y}{\text{Fin}\,(x.y)} \qquad \frac{}{\text{HF}\,\emptyset} \qquad \frac{\text{HF}\,x \qquad \text{HF}\,y}{\text{HF}\,(x.y)} \qquad \frac{x \subseteq \text{WF}}{\text{WF}\,x}$$

$$\frac{}{\text{Num}\,\emptyset} \qquad \frac{\text{Num}\,x}{\text{Num}\,(\sigma x)} \qquad \frac{\mathcal{O}\,x}{\mathcal{O}\,(\sigma x)} \qquad \frac{x \subseteq \mathcal{O}}{\mathcal{O}\,(\bigcup x)}$$

$$\frac{}{\text{BS}\,\emptyset} \qquad \frac{\text{BS}\,x}{\text{BS}\,(\mathcal{P}x)} \qquad \frac{\mathcal{S}\,x}{\mathcal{S}\,(\mathcal{P}x)} \qquad \frac{x \subseteq \mathcal{S}}{\mathcal{S}\,(\bigcup x)}$$

$$\frac{}{\mathcal{T}_0\,\emptyset} \qquad \frac{\mathcal{T}_0\,x}{\mathcal{T}_0\,(gx)} \qquad \frac{\mathcal{T}\,x}{\mathcal{T}\,(gx)} \qquad \frac{x \subseteq \mathcal{T}}{\mathcal{T}\,(\bigcup x)} \qquad \frac{\forall y.\ \mathcal{T}\,y \to y \subset x \to \mathcal{A}\,y}{\mathcal{A}\,x}$$

The classes Num, BS, and HF are small. The corresponding sets can be obtained as $C[\emptyset, \sigma]$, $C[\emptyset, \mathcal{P}]$, and $\bigcup (C[\emptyset, \mathcal{P}])$. The classes Fin, WF, $\mathcal{O}$, and $\mathcal{S}$ are large. Num is a subclass of $\mathcal{O}$ and HF, BS is a subclass of $\mathcal{S}$ and HF, and HF, $\mathcal{O}$, and $\mathcal{S}$ are subclasses of WF. Moreover, $\mathcal{T}_0 \subseteq \mathcal{T} \subseteq \mathcal{A}$.

23 Well-Ordering Theorem and Axiom of Choice

The **axiom of choice** asserts the existence of a **choice function**

$$\begin{aligned} \gamma &: \text{set} \to \text{set} \\ \text{Choice} &: \forall x.\ \text{inhab}\ x \to \gamma x \in x \end{aligned}$$

which for every inhabited set yields an element of the set. The axiom of choice has surprising consequences. One consequence is excluded middle (Diaconescu's Theorem 85). Another consequence is the existence of a well-ordering for every set. Zermelo [14] showed this result first in 1904, before he started axiomatic set theory, assuming a choice function tacitly. Since the real numbers do not have a natural well-ordering, Zermelo's well-ordering theorem was quite a surprise when it was discovered. To provide evidence for his controversial result, Zermelo [15] came up axiomatic set theory to make explicit the assumptions his result relies on.

There are many different formulations of the axiom of choice in the literature. They should all be equivalent. Our formulation is particularly simple.

In the 1908 paper starting axiomatic set theory, Zermelo [15] gave a second proof of the well-ordering theorem that simpler than his first proof from 1904. In contrast to the first proof, the second proof makes the assumptions used for sets explicit.

We will prove the well-ordering theorem in the following. Like Zermelo, we will not assume the axiom of choice but prove a conditional result saying that every set with a choice function can be well-ordered. Our proof is similar to Zermelo's second proof.

Let u be a set. A **well-ordering** of u is a linear ordering such that every class containing an element of u contains a least element that is in u. Formally, a well-ordering of u is a binary predicate $a \le b$ on sets such that the following conditions are satisfied (transitivity, antisymmetry, linearity, existence of least elements).

1. $\forall abc.\ a \in u \to b \in u \to c \in u \to a \le b \to b \le c \to a \le c.$
2. $\forall ab.\ a \in u \to b \in u \to a \le b \to b \le a \to a = b.$
3. $\forall ab.\ a \in u \to b \in u \to a \le b \lor b \le a.$
4. $\forall ap.\ a \in u \to pa \to \exists b.\, b \in u \land pb \land \forall c.\, c \in u \to pc \to b \le c.$

We have already seen some well-orderings. For instance, the inclusion ordering on sets is a well-ordering of every set of ordinals and of every set of stages.

Theorem 163 (Well-Ordering) Let u be a set and γ be a function such that $\gamma x \in x$ for every inhabited $x \subseteq u$. Then one can construct a well-ordering for u.

The proof is interesting. It uses a tower similar to the towers we have seen for the ordinals and the cumulative stages.

Let u be a set and γ be a function such that $\gamma x \in x$ for every inhabited $x \subseteq u$. We refer to γ as **choice function**. We define **complements** as $Cx := \{\, y \in u \mid y \notin x \,\}$ and an **extension function**

$$x^+ \ := \ \{\, z \in u \mid z \in x \lor z = \gamma(Cx) \,\}$$

If x is a proper subset of u, x^+ contains an additional element from u determined by the the choice function γ. We shall rely on the tower $\mathcal{Z}$ generated by x^+:

$$\frac{\mathcal{Z}\,x}{\mathcal{Z}\,(x^+)} \qquad\qquad \frac{x \subseteq \mathcal{Z}}{\mathcal{Z}\,(\bigcup x)}$$

It is easy to see that all stages are subsets of u, and that the generator x^+ is increasing. Thus we have successor linearity and least elements for the stages of $\mathcal{Z}$ (Facts 138 and 146). Thus set inclusion is a well-ordering of $\mathcal{Z}$.

We define an **embedding function**

$$\overline{a} \ := \ \bigcup\{\, x \in \mathcal{P}u \mid \mathcal{Z}\,x \land a \notin x \,\}$$

mapping every set a to the greatest stage not containing a. Using the embedding, we define a binary predicate

$$a \le b \ := \ \overline{a} \subseteq \overline{b}$$

for which we will show that it is a well-ordering of u.

Fact 164 (Transitivity and Linearity) $a \leq b$ is a transitive and linear on u.

Fact 165 (Least Elements)
Every class containing an element of u contains a least such element (wrt $a \leq b$).

Proof Let $a \in u$ and pa. Define $qx := \exists b.\ b \in u \wedge pb \wedge \overline{b} = x$. We have $q(\overline{a})$. By Fact 146 we have some $b \in u$ such that pb and $\overline{b}$ is the least element of $\mathcal{Z}$ satisfying q. It follows that b is the least element of u satisfying p. ■

It remains to show that $a \leq b$ is antisymmetric on u. For this it suffices to show that the tower embedding $\lambda a.\overline{a}$ is injective on u. We establish the injectivity of the embedding by identifying $\lambda x.\ \gamma(Cx)$ as the inverse of the embedding. The proof of this fact is the first and last time we will use the assumption that γ is a choice function for u.

Fact 166 (Inversion) Let $a \in u$. Then $\gamma(C(\overline{a})) = a$.

Proof By contradiction using **excluded middle.** Let $a \in u$ and suppose $\gamma(C(\overline{a})) \neq a$. We have $a \notin \overline{a}$. Thus $\gamma(C(\overline{a})) \in u$ and $\gamma(C(\overline{a})) \notin \overline{a}$ by the assumption that γ is a choice function for u (this is the first and last time we use this assumption). We have $\gamma(C(\overline{a})) \in \overline{a}^+$. We also have $\overline{a}^+ \subseteq \overline{a}$ since $\overline{a}^+$ is a stage not containing a. Contradiction since $\gamma(C(\overline{a})) \in \overline{a}$ and $\gamma(C(\overline{a})) \notin \overline{a}$. ■

Fact 167 (Antisymmetry) $a \leq b$ is antisymmetric on u.

Proof Let $a, b \in u$, $a \leq b$ and $b \leq a$. By extensionality $\overline{a} = \overline{b}$. Thus $a = b$ by Fact 166. ■

Exercise 168 Let a well-ordering for a set u be given. Construct a choice function for u (as specified by Theorem 163).

24 Regularity Axiom

ZFC admits only well-founded sets. This is done with the regularity axiom. Given our development, we could impose the axiom $\forall x.\ \mathrm{WF}\,x$ using the inductive predicate WF. In the usual first-order setting for ZFC inductive predicates are not available. However, there is a simple first-order formulation of the regularity axiom that works provided the infinity axiom is imposed.

A set x is **regular** if in case x is inhabited x has an element y such that x and y have no element in common:

$$\text{regular } x \;:=\; \text{inhab } x \to \exists y \in x\ \forall z \in y.\ z \notin x$$

Fact 169 Every well-founded set is regular.

Proof Follows with Exercise 54 and $pz := z \in x$. ■

A set x is **serial** if it is inhabited and for every element $y \in x$ there exists an element $z \in y$ such that $z \in x$.

Fact 170 A set is serial iff it is not regular.

Proof Follows with **excluded middle**. ■

Fact 171 Let x be a transitive and non-well-founded set. Then $\{ y \in x \mid \neg \mathrm{WF}\ y \}$ is a serial set.

Proof It suffices to show that every non-well-founded set has a non-wellfounded element. This follows with **excluded middle**. ■

Fact 172 There exists a serial set if a non-well-founded exists.

Proof Let x be a non-well-founded set. By Fact 160 we can assume without loss of generality that x is transitive. Thus we have a serial set by Fact 171. ■

Note that the above proof uses the existence of a transitive closure (Fact 160), which in turn relies on the infinity axiom.

Fact 173 Every set is well-founded if and only if every set is regular.

Proof Follows with **excluded middle** and Facts 169, 172, and 170. ■

ZFC's regularity axiom now simply says that every set is regular. We will not impose the regularity axiom since we don't need it.

Exercise 174 (First-Order Characterisation of Ordinals) A set x is **subset regular** if every subset of x is regular. A set x is **linear** if for all $a, b \in x$ either $a \in b$ or $a = b$ or $b \in a$. Prove the following:

a) Every well-founded set is subset regular.

b) Every transitive and subset regular set is well-founded.

c) A set is well-founded iff its transitive closure is subset regular.

d) A set is an ordinal if and only if it is linear, subset regular, and transitive.

25 Appendix: Accessibility and Least Numbers

The bestknown well-ordering is the ordering of the natural numbers, which may be characterised as $x \le y \leftrightarrow \exists z.\ x + z = y$. We will establish accessibility, complete induction, and existence of least elements for this ordering. This is instructive since the notions can be analysed in the familiar setting of numbers and the proofs can be carried out such that they anticipate the general proofs for towers. As it turns out, many of the key ideas of the general proofs appear already in the simplified setting of numbers.

In the following, the letters x, y, z will range over numbers (i.e., elements of the inductive type $\mathbb{N}$) and the letters p, q will range over classes of numbers (i.e., unary predicates on numbers).

Complete induction says that we can prove px by assuming py for all $y < x$. Using complete induction is simpler than using natural induction (the induction principle coming with the inductive definition of $\mathbb{N}$) since the proof goal remains unchanged except for the addition of the inductive hypothesis. Formally, complete induction can be described with the following proposition:

$$\forall p.\ (\forall x.\ (\forall y.\ y < x \to py) \to px) \to \forall x.\ px$$

In type theory, we can capture an induction principle with an inductive predicate we call an *accessibility predicate*. The **accessibility predicate** for complete induction for $\mathbb{N}$ is

$$\frac{\forall y.\ y < x \to \mathcal{A}y}{\mathcal{A}x}$$

We call a number x **accessible** if $\mathcal{A}x$. For every accessible number complete induction is obtained as $\mathcal{A}$-induction. We show that every number is accessible.

Fact 175 (Complete Induction) $\forall x : \mathbb{N}.\ \mathcal{A}x$.

Proof We show $\forall y.\ y \le x \to \mathcal{A}y$ by natural induction on x. It is essential to show this generalised claim.

In the base case we have $y \le 0$ and need to show $\mathcal{A}y$. Since $y = 0$, it suffices to show $\mathcal{A}0$. By unfolding $\mathcal{A}0$, we have $z < 0$ and need to show $\mathcal{A}z$. This is easy since $z < 0$ is contradictory.

In the successor case, we have $y \le Sx$ and need to show $\mathcal{A}y$. By unfolding $\mathcal{A}y$, we have $z < y$ and need to show $\mathcal{A}z$. We have $z \le x$. Now the claim follows by the inductive hypothesis for x. ■

Fact 176 (Least Elements)
Every inhabited class of numbers contains a least element.

Proof Let px. We show $\exists y.\ py \wedge \forall z.\ pz \to y \le z$ by complete induction on x (i.e., by induction on $\mathcal{A}x$). Note that the claim does not depend on x.

Case analysis using **excluded middle.** If x satisfies $\forall z.\ pz \to x \le z$, we have a least element. Otherwise, by **excluded middle**, we have some z such that pz and $x \not\le z$. By linearity we have $z < x$. Now the claim follows by the inductive hypothesis for z. ∎

26 Appendix: Well-Orderings

We may characterise a well-ordering as a linear ordering supporting complete induction. With excluded middle, support of complete induction turns out to be equivalent to the existence of least elements. In the mathematical literature, the characterisation with least elements is used for the definition of a well-ordering. Constructively, the characterisation with complete induction is necessary. As we have just seen, complete induction for numbers can be established constructively and has many constructive applications, while existence of least elements requires excluded middle.

In the following, we explore the two characterisations of a well-ordering for a given binary predicate $x \le y$ on a type X.

We call $x \le y$ a **linear ordering** if it satisfies transitivity, antisymmetry, and linearity:

1. If $x \le y$ and $y \le z$, then $x \le z$.
2. If $x \le y$ and $y \le x$, then $x = y$.
3. Forall x and y, either $x \le y$ or $y \le x$.

Note that linearity implies reflexivity.

We say that $x \le y$ **has least elements** if every inhabited class has a least element: $\forall p.\ \mathrm{ex}\, p \to \exists x.\ px \wedge \forall z.\ pz \to x \le z$.

We define the notation $x < y := x \le y \wedge x \ne y$ and the **accessibility predicate**:

$$\frac{\forall y.\ y < x \to \mathcal{A}y}{\mathcal{A}x}$$

We say that x is **accessible** if $\mathcal{A}x$. We say that $x \le y$ **supports complete induction** if every element is accessible.

Fact 177 If x is inaccessible, then there exists an inaccessible y such that $y < x$.

Proof Let x be inaccessible. We show the claim by contradiction using **excluded middle.** Suppose there does not exist an inaccessible y such that $y < x$. Then, using **excluded middle**, all $y < x$ are accessible. Hence x is accessible. Contradiction. ∎

Fact 178 A linear ordering supports complete induction if and only if it has least elements.

Proof The direction from complete induction to least elements follows with the argument we have used before for numbers (Fact 176). The argument uses **excluded middle** and linearity.

Let a linear ordering with least elements be given. We prove that every element is accessible. Using **excluded middle**, we obtain an inaccessible element and need to exhibit a contradiction. Since the ordering has least elements, there exists a least inaccessible element. This yields a contradiction with Fact 177 and antisymmetry. ∎

Note that the proof does not use transitivity of the given linear ordering.

27 Appendix: Well-Founded Relations

Informally, a relation $x < y$ is *well-founded* if it does not admit an infinite descending chain $\cdots < x_2 < x_1 < x_0$. The most prominent well-founded relation is the ordering $m < n$ of the natural numbers. In contrast to well-orderings, well-founded relations are not required to be transitive or antisymmetric. It will turn out that a linear ordering $x \le y$ is a well-ordering if and only if the strict companion $x < y$ is a well-founded relation.

Well-founded relations appear prominently in set theory. If we impose the regularity axiom, the membership relation is well-founded on all sets. Without the regularity axiom, the membership relation is still well-founded on well-founded sets. In fact, the class of well-founded sets is the greatest class of sets on which membership is well-founded. Strict set inclusion $x \subset y$ is well-founded on the class of ordinals and on the class of cumulative stages. More generally, strict set inclusion $x \subset y$ is well-founded on every tower obtained with an increasing generator.

Well-founded relations are essential in the study of terminating processes and procedures. If a relation $x < y$ is well-founded, a process recursively descending from x to some $y < x$ will always terminate. As it comes to termination proofs, the order of the natural numbers and its lexicographic variants are frequently used.

In the following, we will explore several formal characterisations of well-founded relations in type theory. We start with a type X and a predicate $R : X \to X \to \mathbb{P}$. We call unary predicates on X **classes**.

A class p is **serial** if it is inhabited and for every x in p there is a y in p such that Ryx:

$$\text{serial}\, p \;:=\; \text{ex}\, p \land \forall x.\, px \to \exists y.\, Ryx$$

Serial classes formalize the notion of infinite descending chains. We may define well-founded relations as relations not admitting serial classes.

A **minimal element** of a class p is an x in p such that Ryx for no y in p:

$$\text{minel}\, p\, x \;:=\; px \wedge \forall y.\, py \to \neg Ryx$$

We may define well-founded relations as relations such that every inhabited class has a minimal element. This is the preferred definition in the mathematical literature.

We call a class p **regular** if it has a minimal element if it is inhabited:

$$\text{regular}\, p \;:=\; \text{ex}\, p \to \text{ex}\,(\text{minel}\, p)$$

Fact 179 Every class of X is regular if and only if no class of X is serial.

Proof Suppose every class is regular and p is a serial class. Then it follows constructively that p is inhabited and has no minimal element. Contradiction with the assumption that every class is regular.

Assume no serial class exists. Let p be an inhabited class. We show that p has a minimal element. By **excluded middle** we assume that p has no minimal element. It suffices to show that p is serial. Let px. It suffices to show that Ryx for some y in p. By **excluded middle** we assume that there is no such y. Now it suffices to show that x is a minimal element of p, which is a straightforward consequence of the assumptions collected so far. ■

Well-foundedness of a relation can also be characterised with an induction principle known as **well-founded induction**. It turns out that this characterisation is most useful constructively. We establish the induction principle with an inductive class $\mathcal{A}$ of **accessible elements**:

$$\frac{\forall y.\, Ryx \to \mathcal{A}\, y}{\mathcal{A}\, x}$$

It turns out that a relation on X is well-founded if and only if every element of X is accessible.

Fact 180 If every element of X is accessible, then every class of X is regular.

Proof Let every element of X be accessible and let px. We show by $\mathcal{A}$-induction on x that p has a minimal element. If x is a minimal element of p, we are done. Otherwise, by **excluded middle**, we have a y in p such that Ryx. The claim follows by the inductive hypothesis for y. ■

Fact 181 If no class of X is serial, every element of X is accessible.

Proof Assume that no class is serial and that there is an inaccessible element. It suffices to show that the class of inaccessible elements is serial. Let x be an inaccessible element. We show that there exists an inaccessible element y such that Ryx. By **excluded middle** we assume that no such element exists. It suffices to show that x is accessible. By unfolding the definition of accessibility, we can assume Ryx and prove that y is accessible. By **excluded middle** we assume that y is inaccessible. Contradiction with the assumption introduced by the first use of excluded middle. ∎

Well-founded induction can also be captured without the inductive accessibility predicate. We define the **inductive classes of X** as follows:

$$\text{inductive } p \;:=\; (\forall x.\, (\forall y.\, Ryx \to py) \to px) \to \forall x.\, px$$

Fact 182 Every element of X is accessible if and only if every class of X is inductive.

Proof Straightforward. ∎

Theorem 183 (Well-founded Relations) The following propositions are equivalent:

1. Every element of X is accessible.
2. Every class of X is inductive.
3. Every inhabited class of X has a minimal element.
4. No class of X is serial.

Proof Follows with Facts 179, 180, 181, and 182. ∎

There are some tricky uses of excluded middle in this section. The reader may find peace of mind by browsing through the Coq development where all results have straightforward proofs.

The results in this section generalize results we have seen before for sets: Exercise 54, Fact 106, Exercise 124, Fact 145, Fact 146, Fact 169, Fact 170, and Fact 172. For sets, the well-founded relation is typically the strict version of set inclusion, making it possible to work with least elements rather than minimal elements.

We have formalised infinite descending chains as serial classes. This captures the intuition behind infinite descending chains in so far that every serial class provides the elements for an infinite descending chain. A more faithfull formalisation of infinite descending chains as functions $\mathbb{N} \to X$ has the problem that obtaining a descending chain function from a serial set requires a choice function for the type X.

28 Notes

There are many textbooks on set theory expressing various viewpoints on the subject. Halmos [6] explains essential ideas and results of set theory in ordinary mathematical language addressing the working mathematician. Smullyan and Fitting [13] explain many advanced results of set theory from the perspective of logicians. They use inductive classes to account for numerals and ordinals. Hrbacek and Jech [7] give a detailed mathematical introduction to set theory leaving the logical base implicit. Devlin [3] uses first-order logic as basis of set theory and covers non-well-founded sets.

Set theory originated with the work of Cantor [1, 2] in the 1870s. The first axiomatisation of set theory was given by Zermelo [15, 16] in 1908. Fraenkel [5] contributed the replacement axiom.

Our presentation of the cumulative hierarchy is new in that we define stages without using ordinals (the standard technique uses transfinite induction). Proving successor linearity for stages was difficult at first, but became easy once we discovered Smullyan and Fitting's book [13], which contains a slick successor linearity proof for superinductive classes using the double induction lemma.

There is much more to set theory than what you have seen here. One can show that every well-ordered set is order-isomorphic to exactly one ordinal, and that the class of ordinals and the cumulative hierarchy are order-isomorphic. We recommend Smullyan and Fitting [13] if you would like to know more.

There is a constructive theory of HF sets [12]. One starts from axioms that are different from the ones ZF suggests and defines union, power, separation and replacement. Separation and replacement are restricted to decidable predicates. A model of the axiomatisation can be constructed as a quotient of an inductive type for binary trees. All models of the axiomatisation are isomorphic.

Some set-theoretic results like Zermelo's well-ordering theorem and Zorn's Lemma carry over to general types [11].

References

[1] Georg Cantor. Ueber eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen. *Journal für die reine und angewandte Mathematik*, 77:258–262, 1874.

[2] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen*, 46(4):481–512, 1895.

[3] Keith Devlin. *The Joy of Sets: Fundamentals of Contemporary Set Theory*. Springer, 1993.

[4] Radu Diaconescu. Axiom of choice and complementation. *Proceedings of the American Mathematical Society*, 51(1):176–178, 1975.

[5] Adolf Fraenkel. Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. *Mathematische Annalen*, 86(3):230–237, 1922.

[6] Paul Richard Halmos. *Naive set theory*. Springer, 1960.

[7] Karel Hrbacek and Thomas Jech. *Introduction to Set Theory, Third Edition, Revised and Expanded*. CRC Press, 1999.

[8] Casimir Kuratowski. Sur la notion de l'ordre dans la théorie des ensembles. *Fundamenta mathematicae*, 2(1):161–171, 1921.

[9] John von Neumann. Zur Einführung der transfiniten Zahlen. *Acta Scientiarum Mathematicarum (Szeged)*, 1(4):199–208, 1923.

[10] Bertrand Russell. *The principles of Mathematics*, volume 1. Cambridge University Press, 1903.

[11] Gert Smolka, Steven Schäfer, and Christian Doczkal. Transfinite constructions in classical type theory. In Xingyuan Zhang and Christian Urban, editors, *Interactive Theorem Proving - 6th International Conference, ITP 2015, Nanjing, China, August 24-27, 2015*, LNCS 9236, pages 391–404. Springer, 2015.

[12] Gert Smolka and Kathrin Stark. Hereditarily finite sets in constructive type theory. In Jasmin Christian Blanchette and Stephan Merz, editors, *Interactive Theorem Proving - 7th International Conference, ITP 2016, Nancy, France, August 22-27, 2016*, LNCS 9807, pages 374–390. Springer, 2016.

[13] Raymond M. Smullyan and Melvin Fitting. *Set Theory and the Continuum Problem*. Dover Publications, 2010.

[14] E. Zermelo. Beweis, daß jede Menge wohlgeordnet werden kann. (Aus einem an Herrn Hilbert gerichteten Briefe). *Mathematische Annalen*, 59:514–516, 1904. English translation in van Heijenoort.

[15] Ernst Zermelo. Untersuchungen über die Grundlagen der Mengenlehre. I. *Mathematische Annalen*, 65(2):261–281, 1908. English translation in van Heijenoort.

[16] Ernst Zermelo. Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre. *Fundamenta Mathematicae*, 16:29–47, 1930. English translation in van Heijenoort.