Semantics of Programming Languages
Assignment 8

Andreas Podelski, Patrick Maier, Jan Schwinghammer
http://www.ps.uni-sb.de/courses/sem-ws01/

INFORMATIK

Hand in until January, 16th, 2002

Cartesian Abstraction

Let $D_1, \ldots, D_n$ be n sets and let $D = 2^{D_1 \times \ldots \times D_n}$ be ordered by set inclusion $\subseteq$. For all i, let $\Pi_i : D \to D_i$ be the i-th projection defined by $\Pi_i(S) = \{x_i \mid \langle x_1, \ldots, x_i, \ldots, x_n \rangle \in S\}$. Let $\mathrm{app} : D \to D$ be defined by $\mathrm{app}(S) = \Pi_1(S) \times \ldots \times \Pi_n(S)$.

Exercise 8.1 (6) Prove that there is no function $f : D_1 \times \ldots \times D_n \to D_1 \times \ldots \times D_n$ such that app is the canonical extension of f to sets in the sense that $\mathrm{app}(S) = \{f(s) \mid s \in S\}$.

Let $D^\# = 2^{D_1} \times \ldots \times 2^{D_n}$, ordered by $\sqsubseteq$ which is componentwise set inclusion, i.e., $\langle M_1, \ldots, M_n \rangle \sqsubseteq \langle M'_1, \ldots, M'_n \rangle$ iff $M_i \subseteq M'_i$ for all i. Let $\alpha : D \to D^\#$ be defined by $\alpha(S) = \langle \Pi_1(S), \ldots, \Pi_n(S) \rangle$, and let $\gamma : D^\# \to D$ be defined as $\gamma(\langle M_1, \ldots, M_n \rangle) = M_1 \times \ldots \times M_n$.

Exercise 8.2 (6) Prove that

1. $\mathrm{app} = \gamma \circ \alpha$,
2. for all $S \in D$, $\alpha(S) = \mu \langle M_1, \ldots, M_n \rangle \in D^\# : \gamma(\langle M_1, \ldots, M_n \rangle) \supseteq S$, and
3. for all $\langle M_1, \ldots, M_n \rangle \in D^\#$, $\gamma(\langle M_1, \ldots, M_n \rangle) = \nu S \in D : \alpha(S) \sqsubseteq \langle M_1, \ldots, M_n \rangle$.

Exercise 8.3 (6) Prove that

1. α and γ are monotone,
2. $\gamma \circ \alpha \supseteq \mathrm{id}_D$ and $\alpha \circ \gamma \sqsubseteq \mathrm{id}_{D^\#}$, and
3. for all $S \in D$ and all $\langle M_1, \ldots, M_n \rangle \in D^\#$,
   $S \subseteq \gamma(\langle M_1, \ldots, M_n \rangle) \Longleftrightarrow \alpha(S) \sqsubseteq \langle M_1, \ldots, M_n \rangle$.

Exercise 8.4 (6) Find an abstract domain $D^{\#'}$ and corresponding abstraction and concretization mappings α' and γ' such that $\alpha' \circ \gamma' = \mathrm{id}_{D^{\#'}}$. Of course, all other properties of exercises 8.2 and 8.3 should continue to hold.