Semantics of Programming Languages: Solution of Assignment 2

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Exercise 2.1: (10)

(a) Let M, N be programs and $M =_{op} N$. By definition of $=_{op}$ follows

$$\forall C : C[M] \text{ and } C[N] \text{ are programs} \Rightarrow C[M] \cong C[N]$$

Since M, N are programs, choosing the empty context $\bullet$ yields $M = C[M] \cong C[N] = N$.

(b) Let M, N be programs and $M \cong N$. We prove $C[M] \cong C[N]$ by case analysis.

1. M, N both have normal form K. This yields

$$\begin{array}{l} C[M] \searrow^* \\ \qquad\qquad C[K] \Rightarrow^{(confluence)} C[M] \cong C[N] \\ C[N] \nearrow^* \end{array}$$

2. M, N both diverge. This yields $C[M] \cong C[N]$ by assumption.

Exercise 2.2: (10) Given that

$$(A) \forall C : C[M], C[N] \text{ are progams of type nat } \Rightarrow C[M] \cong C[N]$$

we show that

$$\forall C : C[M], C[N] \text{ are progams of type nat or bool } \Rightarrow C[M] \cong C[N]$$

Let C be context such that $C[M], C[N]$ are programs of type nat or bool.

1. C[M] : nat. $C[M] \cong C[N]$ follows from (A).
2. C[M] : bool. Consider context $C' = \texttt{if}\ []\ \texttt{then 1 else 0}$. Using (A) yields $C'[M] \cong C'[N]$.

   (1) If $C'[M]$ diverges, $C'[N]$ also diverges.

   (2) If $C'[M]$ has normal form, then $C'[N]$ has the same normal form.

   Both cases yield $C[M] \cong C[N]$.

Exercise 2.3: (5)

$$\begin{array}{rl} & F = \lambda f.\ M \\ \texttt{letrec f} = \texttt{M in N} & \rightarrow^* (\lambda f.\ N)\ (\text{fix } F) \\ & \rightarrow^* (\lambda f.\ N)\ (F\ (\text{fix } F)) \\ & \rightarrow^* (\lambda f.\ N)\ (F^2\ (\text{fix } F)) \\ & \rightarrow^* \ldots \\ & \rightarrow^* (\lambda f.\ N)\ (F^n\ (\text{fix } F)) \end{array}$$

No.

Exercise 2.4: (10)

(a)

$$\text{comp} = \lambda f.\, \lambda g.\, \lambda x.\; f\; (g\; x)$$

$$\begin{aligned}\text{comp}\; (\lambda x.\, x \;+\; 1)\; (\lambda x.\, x \;+\; 1)\; 5 &\rightarrow^* (\lambda x.\, (\lambda x.\, x \;+\; 1)\; ((\lambda x.\, x \;+\; 1)\; x))\; 5 \\ &\rightarrow^* (\lambda x.\, x \;+\; 1)\; ((\lambda x.\, x \;+\; 1)\; 5) \\ &\rightarrow^* (\lambda x.\, x \;+\; 1)\; 6 \\ &\rightarrow^* 7\end{aligned}$$

(b)

$$\begin{aligned}\text{not} &= \lambda x.\, \texttt{if x then } \text{false } \texttt{else } \text{true} \\ F &= \lambda f.\, \lambda x.\, \texttt{if x then } \text{true } \texttt{else f } (\text{not } \texttt{x}) \\ R &= \texttt{letrec f} = \lambda \texttt{x.}\, \texttt{if x then } \text{true } \texttt{else f } (\text{not } \texttt{x})\; \texttt{in f } \text{false} \\ R &\rightarrow^* (\lambda f.\; f\; \text{false})\; (\text{fix } F) \\ &\rightarrow^* \texttt{if } \text{false } \texttt{then } \text{true } \texttt{else } (\text{fix } \texttt{F})\; (\text{not } \texttt{false}) \\ &\rightarrow^* (\text{fix } F)\; (\text{not } \; false) \\ &\rightarrow^* \texttt{if } (\text{not false})\; \texttt{then } \text{true } \texttt{else } (\text{fix } \texttt{F})\; (\text{not } (\text{not false})) \\ &\rightarrow^* \text{true}\end{aligned}$$

Exercise 2.5: (5)

$$\lambda x.\, \text{fix } x \rightarrow \lambda x.\, x\; (\text{fix } x)$$
$$\searrow \text{fix}$$

Exercise 2.6: (5)

$$F = \texttt{let f}(\texttt{x}) = \texttt{3 in letrec g}(\texttt{x}) = \texttt{g}(\texttt{x} + \texttt{1})\; \texttt{in f } (\texttt{g 5})$$

$$\begin{aligned}F &\rightarrow^* (\lambda f.\, (\lambda g.\; f\; (g\; 5))\; (\text{fix } \lambda g.\, \lambda x.\, g\; (x \;+\; 1)))\; \lambda x.\, 3 \\ &\rightarrow^* (\lambda g.\, (\lambda x.\, 3)\; (g\; 5))\; (\text{fix } \lambda g.\, \lambda x.\, g\; (x \;+\; 1)) \\ &\rightarrow^* (\lambda x.\, 3)\; (\text{fix } \lambda g.\, \lambda x.\, g\; (x \;+\; 1)) \\ &\rightarrow^* 3\end{aligned}$$

Exercise 2.7: (5)

$$M = \lambda f.\, \lambda x.\, f\ (x\ +\ 1)$$
$$R = \texttt{letrec f(x) = f(x+1) in f 1}$$
$$R \xrightarrow{\text{left}}{}^{*} (\lambda f.\ f\ 1)\ (\text{fix}\ M)$$
$$\xrightarrow{\text{left}}{}^{*} (\text{fix}\ M)\ 1$$

Let $n \in \mathbb{N}$.

$$(\text{fix}\ M)\ n \xrightarrow{\text{left}}{}^{*} M\ (\text{fix}\ M)\ n$$
$$\xrightarrow{\text{left}}{}^{*} (\lambda x.\ (\text{fix}\ M)\ (x\ +\ 1))\ n$$
$$\xrightarrow{\text{left}}{}^{*} (\text{fix}\ M)\ (n\ +\ 1)$$

Hence one can construct an infinite chain issuing from fix $M\ 1$. Since we use $\xrightarrow{\text{left}}$ which is deterministic and complete, this is the only chain possible and R has no normal form.

Exercise 2.8: (5)

$$G = \lambda g.\, \lambda x.\, g\ (x\ +\ 1)$$
$$R = \texttt{let f(x) = 3 in letrec g(x) = g(x+1) in f (g 5)}$$
$$\to^{*} (\lambda g.\ (\lambda x.\ 3)\ (g\ 5))\ (\text{fix}\ G)$$
$$\to^{*} (\lambda g.\ (\lambda x.\ 3)\ (g\ 5))\ (G\ (\lambda x.\ (\text{fix}\ G)\ x))$$
$$\to^{*} (\lambda g.\ (\lambda x.\ 3)\ (g\ 5))\ (\lambda x.\ (\lambda x.\ (\text{fix}\ G)\ x)\ (x\ +\ 1))$$
$$\to^{*} (\lambda x.\ 3)\ (\lambda x.\ (\lambda x.\ (\text{fix}\ G)\ x)\ (x\ +\ 1))\ 5$$
$$\to^{*} (\lambda x.\ 3)\ ((\lambda x.\ (\text{fix}\ G)\ x)\ (5\ +\ 1))$$
$$\to^{*} (\lambda x.\ 3)\ ((\text{fix}\ G)\ 6)$$

Since we use eager reduction which is deterministic, using the same argument as in exercise 2.2.18 shows divergence.

Exercise 2.9: (5)

$$F = \lambda f.\, \lambda y.\, \texttt{if Eq? y 0 then 1 else y} * \texttt{f (y} - \texttt{1)}$$
$$\text{fact} = \text{fix}\ F$$

$$\text{fact}\ 3 \to^{*} (\text{fix}\ F)\ 3$$
$$\to^{*} F\ (\lambda x.\ (\text{fix}\ F)\ x)\ 3$$
$$\to^{*} \texttt{if Eq? 3 0 then 1 else 3} * (\lambda \texttt{x}.\ (\text{fix F})\ \texttt{x})\ (\texttt{3} - \texttt{1})$$
$$\to^{*} 3 * 2 * 1$$
$$\to^{*} 6$$

Exercise 2.10: (5)

$$\text{fix}\ \lambda x.\ x \to \text{fix}\ \lambda x.\ x \to^* \cdots$$
$$\searrow (\lambda x.\ x)\ \lambda x.\ (\text{fix}\ \lambda x.\ x)\ x \to \lambda x.\ (\text{fix}\ \lambda x.\ x)\ x$$

Because fix is substituted below an lambda abstraction, thereby preventing divergence.

Exercise 2.11: (5)

$$\text{search} = \lambda p.\ \text{fix}\ (\lambda f.\ \lambda n.\ \texttt{if p n then n else f}\ (\texttt{n} + \texttt{1}))\ \texttt{0}$$
$$\text{half} = \lambda n.\ search\ (\lambda x.\ \texttt{if Eq?}\ (\texttt{x} + \texttt{x})\ \texttt{n}\ \texttt{then}\ \text{true}\ \texttt{else if Eq?}\ (\texttt{x} + \texttt{x} + \texttt{1})\ \texttt{n then}\ \text{true}\ \texttt{else}\ \text{false})$$

Exercise 2.12: (10)

$$\text{comp} = \lambda n.\ \lambda f.\ \text{fix}\ (\lambda g.\ \lambda i.\ \lambda x.\ \texttt{if Eq? i n then x else g}\ (\text{succ}\ \texttt{i})\ (\texttt{f x}))\ \texttt{0}$$
$$\text{mult} = \lambda \langle m, n \rangle.\ \text{comp}\ m\ (\lambda x.\ x + n)\ 0$$

Exercise 2.13: (20)

(a)

$$\text{prim} = \lambda g.\ \lambda h.\ \lambda n.\ \text{fix}\ (\lambda r.\ \lambda i.\ \lambda x.\ \texttt{if Eq? i n then x else r}\ (\text{succ}\ \texttt{i})\ (\texttt{h}\ \langle \texttt{x}, \texttt{i} \rangle))\ \texttt{0 g}$$

(b)

$$\text{pred} = \lambda n.\ \text{prim}\ 0\ (\lambda \langle x, i \rangle.\ \texttt{if Eq? i 0 then 0 else}\ \text{succ}\ \texttt{x})\ \texttt{n}$$
$$\text{add} = \lambda m.\ \text{prim}\ m\ (\lambda \langle x, i \rangle.\ \text{succ}\ x)$$
$$\text{h} = \lambda \langle \langle x, y \rangle, i \rangle.\ \texttt{if Eq? x 0 then if Eq? y 0 then}\ \langle \texttt{0}, \texttt{0} \rangle\ \texttt{else}\ \langle \texttt{0}, \texttt{1} \rangle\ \texttt{else if Eq? y 0 then}\ \langle \texttt{1}, \texttt{0} \rangle\ \texttt{else}\ \langle \text{pred}\ \texttt{x}, \text{pred}\ \texttt{y} \rangle$$
$$\text{eq} = \lambda x.\ \lambda y.\ \texttt{Eq?}\ (\text{prim}\ \langle \texttt{x}, \texttt{y} \rangle\ \texttt{h x})\ \langle \texttt{0}, \texttt{0} \rangle$$