Semantics of Programming Languages: Solution of Assignment 4

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Exercise 4.1: (20) We have to show that

$$A \text{ trivial} \Leftrightarrow \forall \text{ sort } s \text{ of } \Sigma : A^s = \emptyset \vee \exists a : A^s = \{a\}$$

Proof.

"$\Rightarrow$" Suppose A is trivial, s is a sort of Σ, and $a, b \in A^s$. It suffices to show that $a = b$.

Choose two distinct variables x, y and environment η for Σ, such that $\eta\ x = a$ and $\eta\ y = b$. Since A is trivial, it satisfies $x = y[x : s, y : s]$. Since η satisfies $\{x : s, y : s\}$, we have

$$a = \eta\ x = [\![\, x \,]\!]\eta = [\![\, y \,]\!]\eta = \eta\ y = b.$$

"$\Leftarrow$" Suppose $\forall$ sort s of $\Sigma : A^s = \emptyset \vee \exists A^s = \{a\}$. Let $M = N[\Gamma]$ be a Σ-Equation. Suppose η is an environment for A that satisfies Γ. It suffices to show that $[\![\, M \,]\!]\eta = [\![\, N \,]\!]\eta$.

Let s be the sort of M and N with respect to Σ. By Lemma 3.3.7 we know $[\![\, M \,]\!]\eta \in A^s$ and $[\![\, N \,]\!]\eta \in A^s$. Hence $[\![\, M \,]\!]\eta = [\![\, N \,]\!]\eta$ by the assumption.

Exercise 4.2: (10)

(a)

$$\begin{aligned} P &= f\ x\ y \\ M &= z \\ N &= x \end{aligned}$$

yields

$$\begin{aligned} [M, N/x, y]\ f\ x\ y &= f\ z\ x \\ [M/x]\ ([N/y]\ f\ x\ y) &= f\ z\ z \end{aligned}$$

(b) Consider

$$M_1, \ldots, M_k/x_1, \ldots, x_k\ P$$

and let V be an infinite set of variables. Choosing distinct variables $z_1, \ldots z_k \in V' = V \backslash \{\mathrm{Vars}(M_1), \ldots, \mathrm{Vars}(M_k), \mathrm{Vars}(P)\}$ yields

$$P' = [z_k/x_k] \ldots [z_1/x_1]\ P$$

Since no variables of P' occur in $M_1, \ldots, M_k$, we can freely arrange any of the substitions M_i/z_i, thereby having

$$M_1, \ldots, M_k/x_1, \ldots, x_k\ P = [M_k/z_k] \ldots [M_1/z_1][z_k/x_k] \ldots [z_1/x_1]\ P$$

Exercise 4.3: (10) We have to show that

$$\mathrm{Th}(A) \models M = N[\Gamma] \Rightarrow M = N[\Gamma] \in \mathrm{Th}(A)$$

Suppose that $\mathrm{Th}(A) \models M = N[\Gamma]$. Since $A \models \mathrm{Th}(A) \Rightarrow A \models M = N[\Gamma]$. Thus, $M = N[\Gamma] \in \mathrm{Th}(A)$.

Exercise 4.4: (10)

$$\begin{aligned}\mathcal{E} \text{ semantically consistent} &\Leftrightarrow \exists M = N[\Gamma] : \mathcal{E} \not\models M = N[\Gamma] \\ &\Leftrightarrow \exists A : A \models \mathcal{E} \wedge A \not\models M = N[\Gamma] \\ &\Leftrightarrow A \text{ nontrivial and } A \models \mathcal{E}.\end{aligned}$$

Exercise 4.5: (15)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{M = N[\Gamma, x : s]}{M = N[\Gamma, y : s, x : s]\ y \text{ not in } \Gamma \qquad y = y[\Gamma, y : s]}}{[y/x]\ M = [y/x]\ N[\Gamma, y : s]}}{\vdots}}{[y/x]\ M = [y/x]\ N[\Gamma \cup \Gamma', y : s]} \qquad \dfrac{\dfrac{P = Q[\Gamma]}{\vdots}}{P = Q[\Gamma \cup \Gamma']}}{\dfrac{[P/y][y/x]\ M = [Q/y][y/x]\ N[\Gamma \cup \Gamma']}{[P/x]\ M = [Q/x]\ N[\Gamma \cup \Gamma']}}$$

Exercise 4.6: (15) Let $\Gamma = \{x_1 : s_1, \dots, x_k : s_k\}$ and $S = [M_1, \dots, M_k/x_1, \dots, x_k]$. Using the results from exercise 3.4.9 we have that there exist distinct variables $z_{1,}, \dots, z_k$, none of them occuring in any $M_1, \dots, M_k$, such that

$$S = [M_k/z_k], \dots, [M_1/z_1], [z_k/x_k], \dots [z_1/x_1]$$

This yields

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{N = M[x_1 : s_1, \dots, x_k : s_k] \qquad \overline{z_1 = z_1[z_1 : s_1]}}{[z_1/x_1]\ N = [z_1/x_1]\ M[z_1 : s_1, x_2 : s_2, \dots, x_k : s_k]}}{\vdots}}{[\bar{z}/\bar{x}]\ N = [\bar{z}/\bar{x}]\ M[z_1 : s_1, \dots, z_k : s_k]} \qquad \overline{S(x_1) = S(x_1)[\Gamma']}}{\dfrac{\dfrac{[S(x_1)/z_1][\bar{z}/\bar{x}]\ M = [S(x_1)/z_1][\bar{z}/\bar{x}]\ N[z_2 : s_2, \dots, z_k : s_k, \Gamma']}{\vdots}}{S\ M = S\ N[\Gamma']}}$$

Exercise 4.7: (20)

(a)

$$\begin{aligned}
\text{count } 3 \text{ (insert } 5 \text{ (insert } 3 \text{ empty))} &= \texttt{if Eq? 3 5} \\
&\quad\ \texttt{then (count 3 (insert 3 empty)) + 1} \\
&\quad\ \texttt{else count 3 (insert 3 empty)} \\
&= \text{count } 3 \text{ (insert } 3 \text{ empty)} \\
&= \texttt{if Eq? 3 3 then (count 3 empty) + 1 else (count 3 empty)} \\
&= (\text{count } 3 \text{ empty}) \ + \ 1 \\
&= 0 \ + \ 1 \\
&= 1
\end{aligned}$$

(b) This is a simple proof done by case analysis. Use the algebraic specification to verify each case.

(i) $a = b \wedge a = c$.

(ii) $a = b \wedge a \neq c$.

(iii) $a \neq b \wedge a = c$.

(iv) $a \neq b \wedge a \neq c$.

(c) $A = \langle \text{list}_{\mathbb{N}}, \mathbb{N}, \mathbb{B}, true^A, false^A, 0^A, \ldots, +^A, \text{empty}^A, \text{insert}^A, \text{count}^A, \ldots \rangle$.

$$\begin{aligned}
\text{empty}^A &= \text{nil} \\
\text{insert}^A(x, l) &= \text{cons } x \ l \\
\text{count}^A(n, l) &= \text{cnt}, \text{where}
\end{aligned}$$

$$\text{cnt}(n, l) = \begin{cases} \text{count}^A(n, \text{tl}(\ l)) + 1 & \text{if } n = \text{hd}(l) \\ \text{count}^A(n, \text{tl}(\ l)) & \text{if } n \neq \text{hd}(l) \end{cases}$$

yields

$$\text{insert } x \text{ (insert } y \ m) = \langle x, \langle y, m \rangle \rangle \neq \langle y, \langle x, m \rangle \rangle = \text{insert } y \text{ (insert } x \ m)$$