Semantics of Programming Languages:
Solution of Assignment 5

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Exercise 5.1

(a) Let $M = N[\Gamma] \in \mathrm{Th}(\mathcal{A})$ and h surjective homomorphism. For all η satisfiying Γ in $\mathcal{A}$ we have

$$h(\mathcal{A}[\![M]\!]\eta) = \mathcal{B}[\![M]\!]\eta^h$$
$$h(\mathcal{A}[\![N]\!]\eta) = \mathcal{B}[\![M]\!]\eta^h$$

and thus $\mathcal{B}[\![M]\!]\eta^h = \mathcal{B}[\![N]\!]\eta^h$. Since h is surjective, we have

$$\forall \eta^{'} \text{satisfiying } \Gamma \text{ in } \mathcal{B}\ \exists \eta \text{ such that } \eta^{'} = h \circ \eta$$

This yields $M = N[\Gamma] \in \mathrm{Th}(\mathcal{B})$.

(b) $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Therefore, we have surjective homomorphisms $h : \mathcal{A} \to \mathcal{B}$ and $k : \mathcal{B} \to \mathcal{A}$. (a) yields $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$.

Exercise 5.2 We have to show

$$\exists h : \mathcal{A} \to \mathcal{B} \text{ surjective homomorphism} \Leftrightarrow \mathcal{B} \text{ and } \mathcal{A}/\sim \text{ isomorphic for some } \sim .$$

We define the relation $\sim$ as follows. $a \sim b :\Leftrightarrow h(a) = h(b)$. It is easy to see that $\sim$ is a congruence relation.

"$\Rightarrow$" We proceed with constructing an isomorphism $g : \mathcal{A}/\sim \to \mathcal{B}$ by setting $g([a]) := h(a)$. This is well-defined.

(i) g is surjective because h is surjective.

(ii) g is injective. Let $g([a]) = g([b])$. Then $h(a) = h(b)$. By definition, we have $a \sim b$ and thus $[a] = [b]$.

"$\Leftarrow$" We define $h : \mathcal{A} \to \mathcal{B}$ with $h(a) := g([a])$.

(i) h is homomorphism.

$$\begin{aligned}
h(f(a_1, \ldots, a_n)) &\overset{\text{(def h)}}{=} g([f(a_1, \ldots, a_n)]) \\
&\overset{\text{(def)}}{=} g(f([a_1], \ldots, [a_n])) \\
&\overset{\text{(g hom)}}{=} f(g([a_1]), \ldots, g([a_n])) \\
&\overset{\text{(def h)}}{=} f(h(a_1), \ldots, h(a_n))
\end{aligned}$$

(ii) h is surjective. This follows from the fact that g is a isomorphism and the definition of $\sim$.

Since $\mathcal{A}/\sim$ is isomorphic to itself, "$\Leftarrow$" and exercise 3.5.6 yield $\mathrm{Th}(\mathcal{A}) \subseteq \mathrm{Th}(\mathcal{A}/\sim)$.

Exercise 5.3

(a) Assume that $h : \mathcal{A}_2 \to \mathcal{A}_1$ is homomorphism.

$$h(S^{\mathcal{A}_2}\langle 1,-2\rangle) = h\langle 1,-1\rangle$$
$$h(S^{\mathcal{A}_2}\langle 1,-2\rangle) = S^{\mathcal{A}_1}(h\langle 1,-2\rangle) = h\langle 1,-2\rangle + 1$$

and thus $h\langle 1,-1\rangle > h\langle 1,-2\rangle$. Further we have

$$h(\langle 1,-1\rangle +^{\mathcal{A}_2} \langle 1,-1\rangle) = h\langle 1,-2\rangle$$
$$h(\langle 1,-1\rangle +^{\mathcal{A}_2} \langle 1,-1\rangle) = h\langle 1,-1\rangle +^{\mathcal{A}_1} h\langle 1,-1\rangle = 2 * h\langle 1,-1\rangle$$

and thus $h\langle 1,-1\rangle \leq h\langle 1,-2\rangle$. Therefore, there is no such h and $\mathcal{A}_2$ is not initial algebra.

(b) Assume that $h : \mathcal{A}_3 \to \mathcal{A}_1$ is homomorphism.

$$h(S^{\mathcal{A}_3}[k-2]) = h[k-1]$$
$$h(S^{\mathcal{A}_3}[k-2]) = S^{\mathcal{A}_1}(h[k-2]) = h[k-2] + 1$$

and thus $h[k-1] > h[k-2]$. Further we have

$$h([k-1] +^{\mathcal{A}_3} [k-1]) = h[k-2]$$
$$h([k-1] +^{\mathcal{A}_3} [k-1]) = h[k-1] +^{\mathcal{A}_1} h[k-1]$$

and thus $h[k-1] \leq h[k-2]$. Therefore, there is no such h and $\mathcal{A}_3$ is not initial algebra.

Exercise 5.4

- $x \to M \in \mathcal{R} \Rightarrow x \to_{\mathcal{R}} M \to_{\mathcal{R}} [M/x]\ M \to_{\mathcal{R}} \ldots$ since $M \to_{\mathcal{R}} [M/x]\ M$ is instance of $x \to_{\mathcal{R}} M$.
- $N \to_{\mathcal{R}} M$ where x occurs in M but not in N. Setting $x = N$ yields $x \to_{\mathcal{R}} N \to_{\mathcal{R}} [N/x]\ N \to_{\mathcal{R}} \ldots$

Exercise 5.5

$$\begin{aligned} \mathcal{A} &= \langle \mathbb{N}\backslash\{0,1\}, \mathrm{or}^{\mathcal{A}}, \mathrm{and}^{\mathcal{A}}, \mathrm{not}^{\mathcal{A}} \rangle \\ \mathrm{or}^{\mathcal{A}}(x,y) &= x * y \\ \mathrm{and}^{\mathcal{A}}(x,y) &= x + y + 1 \\ \mathrm{not}^{\mathcal{A}}(x) &= 2^x \end{aligned}$$

Exercise 5.6

(a) Let $\mathcal{A} = \langle \mathbb{N}\backslash\{0,1\}, 0^{\mathcal{A}}, \mathrm{S}^{\mathcal{A}}, +^{\mathcal{A}}\rangle$.

$$\begin{aligned} 0^{\mathcal{A}} &= 2 \\ \mathrm{S}^{\mathcal{A}}(x) &= x+2 \\ +^{\mathcal{A}}(x,y) &= x*y \end{aligned}$$

(b) Let $\mathcal{A} = \langle \mathbb{N}\backslash\{0,1\}, 0^{\mathcal{A}}, \mathrm{S}^{\mathcal{A}}, +^{\mathcal{A}}, *^{\mathcal{A}}\rangle$.

$$\begin{aligned} 0^{\mathcal{A}} &= 2 \\ \mathrm{S}^{\mathcal{A}}(x) &= x+2 \\ +^{\mathcal{A}}(x,y) &= x*y \\ *^{\mathcal{A}}(x,y) &= y^x \end{aligned}$$

Exercise 5.7

$(3+1)$

$$(0+y)+z \xrightarrow{1} y+z$$
$$\overset{3}{\searrow} 0+(y+z) \xrightarrow{1} y+z$$

$(3+2)$

$$((-x)+x)+z \xrightarrow{2} 0+z \xrightarrow{1} z\ (NF)$$
$$\overset{3}{\searrow} (-x)+(x+z)\ (NF) \text{ not joinable}$$

$(3+3)$

$$((x+y)+z)+a \xrightarrow{3} (x+y)+(z+a) \xrightarrow{3} x+(y+(z+a))$$
$$\overset{3}{\searrow} (x+(y+z))+a \xrightarrow{3} x+(((y+z)+a) \xrightarrow{3} x+(y+(z+a))$$

Exercise 5.8

$(1+1)$

$$\neg\neg\neg x \xrightarrow{1} \neg x$$
$$\overset{1}{\searrow} \neg x$$

$(1+2)$

$$\neg\neg(x \vee y) \xrightarrow{1} x \vee y$$
$$\overset{2}{\searrow} \neg(\neg x \wedge \neg y) \xrightarrow{3} \neg\neg x \vee \neg\neg y \xrightarrow{1,1} x \vee y$$

$(1+3)$

$$\neg\neg(x \wedge y) \xrightarrow{1} x \wedge y$$
$$\overset{3}{\searrow} \neg(\neg x \vee \neg y) \xrightarrow{2} \neg\neg x \wedge \neg\neg y \xrightarrow{1,1} x \wedge y$$

$(3+4)$

$$\neg(x \wedge (y \vee z)) \xrightarrow{3} \neg x \vee \neg(y \vee z) \xrightarrow{2} \neg x \vee (\neg y \wedge \neg z)\ (NF)$$
$$\overset{4}{\searrow} \neg((x \wedge y) \vee (x \wedge z)) \xrightarrow{2} \neg(x \wedge y) \wedge \neg(x \wedge z) \xrightarrow{3,3} (\neg x \vee \neg y) \wedge (\neg x \vee \neg z)$$
$$\xrightarrow{4} ((\neg x \vee \neg y) \wedge \neg x) \vee ((\neg x \vee \neg y) \wedge \neg z)$$
$$\xrightarrow{5,5} ((\neg x \wedge \neg x) \vee (\neg y \wedge \neg x)) \vee ((\neg x \wedge \neg z) \vee (\neg y \wedge \neg z))\ (NF) \text{ not joinable}$$

$(3+5)$ analogously

$(4+5)$

$$(x \vee y) \wedge (z \vee a) \xrightarrow{4} ((x \vee y) \wedge z) \vee ((x \vee y) \wedge a) \xrightarrow{5,5} ((x \wedge z) \vee (y \wedge z)) \vee ((x \wedge a) \vee (y \wedge a))\ (NF)$$
$$\xrightarrow{5} (x \wedge (z \vee a)) \vee (y \wedge (z \vee A))$$
$$\xrightarrow{4,4} ((x \wedge z) \vee (x \wedge a)) \vee ((y \wedge z) \vee (y \wedge a))\ (NF) \text{ not joinable}$$

Further we have $(5+4) = (4+5)$.