Semantics of Programming Languages:
Solution of Assignment 6

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Exercise 6.1: (10)

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\sigma \triangleright x:\sigma}\;(\text{var})}{\emptyset \triangleright \lambda x:\sigma.x : \sigma \to \sigma}\;(\to\ \text{intro})}{y:\sigma \triangleright \lambda x:\sigma.x : \sigma \to \sigma}\;(\text{add var}) \qquad \dfrac{}{y:\sigma \triangleright y:\sigma}\;(\text{var})}{y:\sigma \triangleright (\lambda x:\sigma.x)\ y : \sigma}\;(\to\ \text{elim})$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\sigma \triangleright x:\sigma}\;(\text{var})}{y:\sigma, x:\sigma \triangleright x:\sigma}\;(\text{add var})}{y:\sigma \triangleright (\lambda x:\sigma.x) : \sigma \to \sigma}\;(\to\ \text{Intro}) \qquad \dfrac{}{y:\sigma \triangleright y:\sigma}\;(\text{var})}{y:\sigma \triangleright (\lambda x:\sigma.x)\ y : \sigma}\;(\to\ \text{elim})$$

Exercise 6.2: (5) The Σ-Signature

sorts bool

fctns

$$a, b : \text{bool}$$
$$f : \text{bool} \times \text{bool} \to \text{bool}$$

becomes the $\Sigma_{\to}$ Signature

type cns bool

term cns

$$a, b : \text{bool}$$
$$f : \text{bool} \to \text{bool} \to \text{bool}$$

With appropriate Γ, we have that $f\ a\ b \in \text{Terms}^{\text{bool}}(\Sigma, \Gamma)$ and $f\ a\ b \in \text{Terms}(\Sigma_{\to})$. Further, $fa : \text{bool} \to \text{bool}$ well-formed in $\Sigma_{\to}$ and $fa \notin \text{Terms}^{\text{bool}}(\Sigma, \Gamma)$.

Exercise 6.3: (10)

$$\lambda x : a.x : a \to a$$

$$\lambda x : a \to b.\lambda y : c.x : (a \to b) \to c \to a \to b$$

$$\lambda f_1 : a \to b.\lambda f_2 : b \to c.\lambda x : a.f_2\ (f_1\ x) : (a \to b) \to (b \to c) \to (a \to c)$$

Exercise 6.4: (40)

```
(* (a) ((a -> null) + (b -> null)) -> (a x b -> null) *)
fn x : ('a -> null, 'b -> null) plus =>
fn y => Case x (fn f => f (P1 y)) (fn f => f (P2 y));

(* (b) (a -> null) x (b -> null) -> (a + b -> null) *)
fn x : ('a -> null) * ('b -> null) => fn y =>
Case y (fn y => (P1 x) y) (fn y => (P2 x) y);

(* (c) ((a -> c) + (b -> c)) -> a x b -> c *)
fn x => fn y : 'a * 'b => Case x (fn f => f (P1 y)) (fn f => f (P2 y));

(* (d) ((a x b) + (a x c)) -> (a x (b + c)) *)
fn x => Case x (fn ab => (P1 ab, L (P2 ab))) (fn ac => (P1 ac, R (P2 ac)));

(* (e) ((a -> b) x (a -> b -> null)) -> a -> null *)
fn x : ('a -> 'b) * ('a -> 'b -> null) =>
fn a => (fn b => (P2 x) a b) ((P1 x) a);

(* (f) ((a -> null) + b) -> a -> b *)
fn x => dneg (fn f => Case x
(fn g => f (fn x => zero (g x))) (fn b => f (fn a => b)));

(* (g) (((a x (b -> null)) -> c) x (c -> null)) x a -> b *)
fn x => dneg (fn f => (P2 (P1 x)) ((fn a => (P1 (P1 x)) (a, f)) (P2 x)));

(* (h) (a + (a -> null)) *)
dneg (fn x => x (R (fn y => x (L y))));

(* (i) ((a + b) -> null) -> (a -> null) x (b -> null) *)
(fn f : ('a, 'b) S.plus -> null => (fn y => f (L y), fn y => f (R y)));
```

Exercise 6.5: (15)

type cns i, t

term cns

$$\begin{aligned} a, b &: i \\ f &: i \to i \\ g &: i \to i \to i \\ p &: i \to t \\ q &: i \to i \to t \\ \text{false} &: t \\ \text{not} &: t \to t \\ \text{and} &: t \to t \to t \\ \text{exists}_i &: (i \to t) \to t \end{aligned}$$

$$\text{and (not false) (exists}_i\ \lambda x.\text{exists}_i\ \lambda y.(\text{and}\ (p\ (f\ x))(q\ a\ (g\ y\ x))))$$

Exercise 6.6: (20)

(a)

$$\frac{}{x : \sigma \rhd x : \sigma} \quad (\text{var}) \qquad \frac{\Gamma \rhd M : \sigma}{\Gamma, x : \sigma \rhd M : \sigma} \quad (\text{add var})$$

$$\frac{\Gamma, x : \sigma \rhd M : \tau}{\Gamma \rhd (\lambda x : \sigma.M) : \sigma \to \tau} \quad (\to\ \text{Intro}) \qquad \frac{\Gamma \rhd M : \sigma \to \tau, \Gamma \rhd N : \sigma}{\Gamma \rhd M\ N : \tau} \quad (\to\ \text{Elim})$$

$$\frac{\Gamma \rhd M : \sigma, \Gamma \rhd N : \tau}{\Gamma \rhd \langle M, N \rangle : \sigma \times \tau} \quad (\times\ \text{Intro})$$

$$\frac{\Gamma \rhd M : \sigma \times \tau}{\Gamma \rhd \mathbf{Proj}_1^{\sigma,\tau} M : \sigma} \quad (\times\ \text{Elim})_1 \qquad \frac{\Gamma \rhd M : \sigma \times \tau}{\Gamma \rhd \mathbf{Proj}_2^{\sigma,\tau} M : \tau} \quad (\times\ \text{Elim})_2$$

$$\frac{\Gamma \rhd M : \sigma}{\Gamma \rhd \mathbf{Inleft}^{\sigma,\tau} M : \sigma + \tau} \quad (+\ \text{Intro})_1 \qquad \frac{\Gamma \rhd M : \tau}{\Gamma \rhd \mathbf{Inright}^{\sigma,\tau} M : \sigma + \tau} \quad (+\ \text{Intro})_2$$

$$\frac{\Gamma \rhd M : \sigma + \tau, \Gamma \rhd N : \sigma \to \rho, \Gamma \rhd P : \tau \to \rho}{\Gamma \rhd \mathbf{Case}^{\sigma,\tau,\rho}\ M\ N\ P : \rho} \quad (+\ \text{Elim})$$

$$\frac{}{* : \text{unit}} \quad (\text{unit Intro})$$

$$\frac{}{\mathbf{Zero}^{\sigma} : \text{null} \to \sigma} \quad (\text{null Elim})$$

(b)

$$\frac{\Gamma \triangleright M = N : \sigma}{\Gamma, x : \sigma \triangleright M = N : \sigma} \quad (\text{add var}) \qquad \frac{}{\Gamma \triangleright M = M : \sigma} \quad (\text{ref})$$

$$\frac{\Gamma \triangleright M = N : \sigma}{\Gamma \triangleright N = M : \sigma} \quad (\text{sym}) \qquad \frac{\Gamma \triangleright M = N : \sigma, \Gamma \triangleright N = P : \sigma}{\Gamma \triangleright M = P : \sigma} \quad (\text{trans})$$

$$\frac{\Gamma, x : \sigma \triangleright M = N : \tau}{\Gamma \triangleright \lambda x : \sigma.M = \lambda x : \sigma.N : \sigma \to \tau} \quad (\xi) \qquad \frac{\Gamma \triangleright M_1 = M_2 : \sigma \to \tau, \Gamma \triangleright N_1 = N_2 : \sigma}{\Gamma \triangleright M_1\ N_1 = M_2\ N_2 : \tau} \quad (\nu)$$

$$\frac{}{\Gamma \triangleright \lambda x : \sigma.M = \lambda y : \sigma.[y/x]\ M : \sigma \to \tau, \text{ provided } y \notin FV(M)} \quad (\alpha)$$

$$\frac{}{\Gamma \triangleright (\lambda x : \sigma.M)\ N = [N/x]\ M : \tau} \quad (\beta)$$

$$\frac{}{\Gamma \triangleright \lambda x : \sigma.(M\ x) = M : \sigma \to \tau, \text{ provided } x \notin FV(M)} \quad (\eta)$$

$$\frac{}{\Gamma \triangleright \mathbf{Proj_1}\langle M, N\rangle = M : \sigma} \quad (\text{proj}_1) \qquad \frac{}{\Gamma \triangleright \mathbf{Proj_2}\langle M, N\rangle = N : \sigma} \quad (\text{proj}_2)$$

$$\frac{}{\Gamma \triangleright \langle \mathbf{Proj}_1 M, \mathbf{Proj}_2 M\rangle = M : \sigma} \quad (\text{sp})$$

$$\frac{}{\Gamma \triangleright \mathbf{Case}^{\sigma,\tau,\rho}(\mathbf{Inleft}^{\sigma,\tau}\ M)\ N\ P = N\ M : \rho} \quad (\text{case}_1)$$

$$\frac{}{\Gamma \triangleright \mathbf{Case}^{\sigma,\tau,\rho}(\mathbf{Inright}^{\sigma,\tau}\ M)\ N\ P = P\ M : \rho} \quad (\text{case}_2)$$

$$\frac{}{\Gamma \triangleright \mathbf{Case}^{\sigma,\tau,\rho}\ M\ (N \circ \mathbf{Inleft}^{\sigma,\tau})\ (N \circ \mathbf{Inright}^{\sigma,\tau}) = NM : \sigma + \tau} \quad (\text{case}_3)$$

$$\frac{}{\Gamma M = * : \text{unit}}$$

$$\frac{}{\Gamma \triangleright M = \mathbf{Zero}^{\sigma} : \text{null} \to \sigma} \quad (\text{null})$$