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Errata. The projections Π_i should be functions from $D = 2^{D_1 \times \ldots \times D_n}$ to 2^{D_i}, otherwise their definitions would be nonsense.

Exercise 8.1 To prove that in general there is no such function f it suffices to show that there is none for $n = 2$ and $D_1 = D_2 = \{0, 1\}$.

Let $S = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$. Then $\text{app}(S) = \{0, 1\} \times \{0, 1\}$ is a set of cardinality 4 whereas for every function $f : D_1 \times D_2 \to D_1 \times D_2$, the set $\{f(s) \mid s \in S\} = \{f(\langle 0, 0\rangle), f(\langle 1, 1\rangle)\}$ has cardinality 1 or 2.

Exercise 8.2

1. Let $S \in D$. Then $\text{app}(S) = \Pi_1(S) \times \ldots \times \Pi_n(S) = \gamma(\langle \Pi_1(S), \ldots, \Pi_n(S)\rangle) = \gamma(\alpha(S))$.

2. Let $S \in D$. To prove that $\alpha(S)$ is the least element in $D^\#$ whose image under γ is above S, fix an arbitrary $\langle M_1, \ldots, M_n\rangle \in D^\#$ such that $\gamma(\langle M_1, \ldots, M_n\rangle) \supseteq S$. Let $x \in \Pi_i(S)$, i. e., there exists $\langle x_1, \ldots, x_i, \ldots, x_n\rangle \in S$ such that $x = x_i$. Then $\langle x_1, \ldots, x_i, \ldots, x_n\rangle \in \gamma(\langle M_1, \ldots, M_i, \ldots, M_n\rangle) = M_1 \times \ldots \times M_i \times \ldots \times M_n$, so $x = x_i \in M_i$. Hence $\alpha(S) = \langle \Pi_1(S), \ldots, \Pi_n(S)\rangle \sqsubseteq \langle M_1, \ldots, M_n\rangle$. Furthermore $\gamma(\alpha(S)) \supseteq S$, see exercise 8.3.2.

3. Let $\langle M_1, \ldots, M_n\rangle \in D^\#$. To prove that $\gamma(\langle M_1, \ldots, M_n\rangle)$ is the greatest element in D whose image under α is below $\langle M_1, \ldots, M_n\rangle$, fix an arbitrary $S \in D$ such that $\alpha(S) \sqsubseteq \langle M_1, \ldots, M_n\rangle$. Since $\langle \Pi_1(S), \ldots, \Pi_n(S)\rangle = \alpha(S) \sqsubseteq \langle M_1, \ldots, M_n\rangle$, we have for all i, $\Pi_i(S) \subseteq M_i$, so $\Pi_1(S) \times \ldots \times \Pi_n(S) \subseteq M_1 \times \ldots \times M_n$. Hence $S \subseteq \Pi_1(S) \times \ldots \times \Pi_n(S) \subseteq M_1 \times \ldots \times M_n = \gamma(\langle M_1, \ldots, M_n\rangle)$. Furthermore $\alpha(\gamma(\langle M_1, \ldots, M_n\rangle)) \sqsubseteq \langle M_1, \ldots, M_n\rangle$, see exercise 8.3.2.

Exercise 8.3

1. To prove monotonicity of α, let $S, S' \in D$ such that $S \subseteq S'$. Let $x \in \Pi_i(S)$, i. e., there exists $\langle x_1, \ldots, x_i, \ldots, x_n\rangle \in S$ such that $x = x_i$. Then $\langle x_1, \ldots, x_i, \ldots, x_n\rangle \in S'$, so $x = x_i \in \Pi_i(S')$. Hence for all i, $\Pi_i(S) \subseteq \Pi_i(S')$, and thus $\alpha(S) = \langle \Pi_1(S), \ldots, \Pi_n(S)\rangle \sqsubseteq \langle \Pi_1(S'), \ldots, \Pi_n(S')\rangle = \alpha(S')$.

   To prove monotonicity of γ, let $\langle M_1, \ldots, M_n\rangle, \langle M'_1, \ldots, M'_n\rangle \in D^\#$ such that $\langle M_1, \ldots, M_n\rangle \sqsubseteq \langle M'_1, \ldots, M'_n\rangle$. Then $M_i \subseteq M'_i$ for all i, hence $\gamma(\langle M_1, \ldots, M_n\rangle) = M_1 \times \ldots \times M_n \subseteq M'_1 \times \ldots \times M'_n = \gamma(\langle M'_1, \ldots, M'_n\rangle)$.

2. To prove $\text{id}_D \subseteq \gamma \circ \alpha$, let $S \in D$. If $\langle x_1, \ldots, x_n\rangle \in S$ then $x_i \in \Pi_i(S)$ for all i, so $\langle x_1, \ldots, x_n\rangle \in \Pi_1(S) \times \ldots \times \Pi_n(S) = \gamma(\langle \Pi_1(S), \ldots, \Pi_n(S)\rangle) = \gamma(\alpha(S))$. Hence $S \subseteq \gamma(\alpha(S))$.

To prove $\alpha \circ \gamma \sqsubseteq \mathrm{id}_{D^\#}$, let $\langle M_1, \ldots, M_n \rangle \in D^\#$. Then $\alpha(\gamma(\langle M_1, \ldots, M_n \rangle)) = \alpha(M_1 \times \ldots \times M_n) = \langle \Pi_1(M_1 \times \ldots \times M_n), \ldots, \Pi_n(M_1 \times \ldots \times M_n) \rangle$. Now we have to distinguish two cases. If $M_1 \times \ldots \times M_n = \emptyset$ then for all i, $\Pi_i(M_1 \times \ldots \times M_n) = \emptyset$ otherwise $\Pi_i(M_1 \times \ldots \times M_n) = M_i$. Thus, $\alpha(\gamma(\langle M_1, \ldots, M_n \rangle))$ either equals $\langle M_1, \ldots, M_n \rangle$ or $\langle \emptyset, \ldots, \emptyset \rangle$. Hence in any case, $\alpha(\gamma(\langle M_1, \ldots, M_n \rangle)) \sqsubseteq \langle M_1, \ldots, M_n \rangle$.

3. Let $S \in D$ and $\langle M_1, \ldots, M_n \rangle \in D^\#$. Then

$$\begin{aligned}
S \subseteq \gamma(\langle M_1, \ldots, M_n \rangle) &\Leftrightarrow S \subseteq M_1 \times \ldots \times M_n \\
&\Leftrightarrow \forall i \in \{1, \ldots, n\}\ \forall \langle x_1, \ldots, x_i, \ldots, x_n \rangle \in S : x_i \in M_i \\
&\Leftrightarrow \forall i \in \{1, \ldots, n\} : \Pi_i(S) \subseteq M_i \\
&\Leftrightarrow \langle \Pi_1(S), \ldots, \Pi_n(S) \rangle \sqsubseteq \langle M_1, \ldots, M_n \rangle \\
&\Leftrightarrow \alpha(S) \sqsubseteq \langle M_1, \ldots, M_n \rangle \ .
\end{aligned}$$

Exercise 8.4 Set $D^{\#'} = \{\langle \emptyset, \ldots, \emptyset \rangle\} \cup \{\langle M_1, \ldots, M_n \rangle \in 2^{D_1 \times \ldots \times D_n} \mid M_i \neq \emptyset \text{ for all } i\}$. Abstraction and concretization mappings are as before, i.e., $\alpha'(S) = \langle \Pi_1(S), \ldots, \Pi_n(S) \rangle$ and $\gamma'(\langle M_1, \ldots, M_n \rangle) = M_1 \times \ldots \times M_n$, hence the properties demanded in exercises 8.2 and 8.3 continue to hold. To see why $\alpha' \circ \gamma' = \mathrm{id}_{D^{\#'}}$ recall the case distinction in the proof of $\alpha \circ \gamma \sqsubseteq \mathrm{id}_{D^\#}$.