# Polymorphic Lambda Calculus F

## Term-Type Presentation

$X \in TVar$
$T \in Ty \;=\; X \;\mid\; T \to T \;\mid\; \forall X.T$
$x \in Var$
$t \in Ter \;=\; x \;\mid\; \lambda x{:}T.t \;\mid\; t\,t \;\mid\; \lambda X.t \;\mid\; t\,T$
$\Gamma \in Var \rightharpoonup Ty$

$$\frac{\Gamma x = T}{\Gamma \vdash x : T} \qquad \frac{\Gamma[x := T] \vdash t : T'}{\Gamma \vdash \lambda x{:}T.t : T \to T'} \qquad \frac{\Gamma \vdash t_1 : T \to T' \quad \Gamma \vdash t_2 : T}{\Gamma \vdash t_1\, t_2 : T'}$$

$$\frac{\Gamma \vdash t : T \quad X \text{ not free in } \Gamma}{\Gamma \vdash \lambda X.t : \forall X.T} \qquad \frac{\Gamma \vdash t : \forall X.T'}{\Gamma \vdash t\,T : T'[X := T]}$$

## Uniform Presentation

$x \in Var$
$e \in Exp \;=\; x \;\mid\; \lambda x{:}e.e \;\mid\; e\,e \;\mid\; \Pi x{:}e.e \;\mid\; \star \;\mid\; \Box$
$\Gamma \in Var \rightharpoonup Exp$

$$\begin{aligned} \forall X.T &\rightsquigarrow \Pi X{:}\star.T \\ T \to T' &\rightsquigarrow \Pi X{:}T.T' \\ \lambda X.t &\rightsquigarrow \lambda X{:}\star.t \end{aligned}$$

Define **sorting relation** $\Gamma \vdash e : e'$ such that:

- $e$ **type** $\iff$ $\exists \Gamma\colon\; \Gamma \vdash e : \star$
- $e$ **term** $\iff$ $\exists \Gamma, e'\colon\; \Gamma \vdash e : e' \;\wedge\; \Gamma \vdash e' : \star$
- **Sort Hierarchy** term : type : $\star$ : $\Box$

Will use **suggestive Metavariables**:

- $e$ for terms and types
- $s$ for types and $\star$ (sorts)
- $k$ for $\star$ and $\Box$ (kinds)

$\Gamma,\; x : s \;\stackrel{\text{def}}{=}\; \Gamma \cup \{(x, s)\}$ where $x \notin Dom\,\Gamma$

$$\frac{}{\Gamma,\ x:s \vdash x:s} \qquad \frac{}{\Gamma \vdash \star:\Box}$$

$$\frac{\Gamma \vdash s:k \quad k \in \{\star,\Box\} \quad \Gamma,\ x:s \vdash s':\star}{\Gamma \vdash \Pi x{:}s.s':\star}$$

$$\frac{\Gamma \vdash e_1 : \Pi x{:}s.s' \quad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1\, e_2 : s'[x := e_2]}$$

$$\frac{\Gamma \vdash s:k \quad \Gamma,\ x:s \vdash e:s' \quad s'' = \Pi x{:}s.s' \quad \Gamma \vdash s'':k'}{\Gamma \vdash \lambda x{:}s.e : s''}$$

$\Gamma$ **well-formed**:

- $\emptyset$ well-formed
- if $\Gamma$ well-formed and $\Gamma \vdash s:k$, then $\Gamma, x:s$ well-formed

## Uniform de Bruijn Presentation

$n \in Var = \mathbb{N}$

$e \in Exp \;=\; n \;\mid\; \lambda e.e \;\mid\; e\,e \;\mid\; \Pi e.e \;\mid\; \star \;\mid\; \Box$

$\Gamma \in \mathcal{L}(Exp)$

Notation: $\Gamma = [e_1, \ldots, e_n]$, $\Gamma.i = e_i$

$$\frac{s = up\ (n+1)\ (\Gamma.(n+1))}{\Gamma \vdash n:s} \qquad \frac{}{\Gamma \vdash \star:\Box}$$

$$\frac{\Gamma \vdash s:k \quad k \in \{\star,\Box\} \quad s :: \Gamma \vdash s':\star}{\Gamma \vdash \Pi s.s':\star}$$

$$\frac{\Gamma \vdash e_1 : \Pi s.s' \quad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1\, e_2 : \beta\, s' e_2}$$

$$\frac{\Gamma \vdash s:k \quad s :: \Gamma \vdash e:s' \quad s'' = \Pi s.s' \quad \Gamma \vdash s'':k'}{\Gamma \vdash \lambda s.e : s''}$$