Proof of Substitution Property, case “t abstraction”

First, we need three propositions:

P1 $\Gamma \vdash t : T \;\wedge\; x \notin \mathit{Dom}\,\Gamma \;\Longrightarrow\; \Gamma, x : T' \vdash t : T$

P2 $\Gamma \vdash t : T \;\Longrightarrow\; \mathit{FV}t \subseteq \mathit{Dom}\,\Gamma$

P3 $(\lambda x{:}T.t)[x' := t'] \;=\; \lambda x{:}T.t[x' := t']$ if $x \neq x'$ and $x \notin \mathit{FV}t'$

Prove P1 and P2 for exercise, they follow by induction on $|t|$.

Claim $\forall \Gamma, x, t, t', T, T'$: $\Gamma, x : T' \vdash t : T \;\wedge\; \Gamma \vdash t' : T' \;\Longrightarrow\; \Gamma \vdash t[x := t'] : T$

Proof By induction on $|t|$.
Let $\Gamma, x : T' \vdash t : T$ and $\Gamma \vdash t' : T'$. (1)
Let t be an abstraction (case considered).
By definition of typing relation:
$T = T_1 \to T_2$ (for some T_1, T_2)
$t = \lambda x_1 : T_1.\ t_1$ and $x_1 \neq x$ and $x_1 \notin \mathit{Dom}\,\Gamma$ (for some x_1, t_1) (2)
$\Gamma,\ x : T',\ x_1 : T_1 \vdash t_1 : T_2$
By P1, induction hypothesis, and definition of typing relation:
$\Gamma,\ x_1 : T_1 \vdash t_1[x := t'] : T_2$
$\Gamma \vdash \lambda x_1 : T_1.\ t_1[x := t'] : T$
By (2), (1) and P2:
$x_1 \neq x$ and $x_1 \notin \mathit{FV}t'$. By P3 and (2):
$\Gamma \vdash (\lambda x_1 : T_1.\ t_1)[x := t'] : T$
$\Gamma \vdash t[x := t'] : T$ $\square$