Subtyping

A **subtype relation** is a partial order $\leq$ on the set of types that satisfies the **subsumption property**

$$\Gamma \vdash t : T \;\wedge\; \Gamma' \leq \Gamma \;\wedge\; T \leq T' \;\Longrightarrow\; \Gamma' \vdash t : T'$$

where

$$\Gamma \leq \Gamma' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathit{Dom}\,\Gamma = \mathit{Dom}\,\Gamma' \;\wedge\; \forall X \in \mathit{Dom}\,\Gamma\colon\; \Gamma x \leq \Gamma' x$$

Type systems with a subtype relation generalize ordinary type systems, which are obtained by taking the identity relation as subtype relation. In place of the Unique Type Property we now require the more general **Least Type Property**:

$$\{\, T \mid \Gamma \vdash t : T \,\} \neq \emptyset \;\Longrightarrow\; \{\, T \mid \Gamma \vdash t : T \,\} \text{ has a least element}$$

As always, we are only interested in typing relations that have the Preservation and the Progress Property. The **strong typing relation** $\mapsto$ is characterized as follows:

$$\Gamma \mapsto t : T \;\Longleftrightarrow\; T \text{ is the least element of } \{\, T' \mid \Gamma \vdash t : T' \,\}$$

Proposition 1 $\quad \Gamma \vdash t : T \;\Longleftrightarrow\; \exists T'\colon\; \Gamma \mapsto t : T' \;\wedge\; T' \leq T$

STR

STR is a simply typed lambda calculus with records and subtyping defined as follows:

$$
\begin{aligned}
l &\in \mathit{Lab} \\
F &\in \mathit{Lab} \overset{\mathit{fin}}{\rightharpoonup} \mathit{Ty} \\
T &\in \mathit{Ty} \;=\; \mathrm{Top} \;\mid\; T \to T \;\mid\; \Pi F \\
x &\in \mathit{Var} \\
f &\in \mathit{Lab} \overset{\mathit{fin}}{\rightharpoonup} \mathit{Ter} \\
t &\in \mathit{Ter} \;=\; x \;\mid\; \lambda x{:}T.t \;\mid\; t\,t \;\mid\; \pi f \;\mid\; t.l \\
\Gamma &\in \mathit{Var} \rightharpoonup \mathit{Ty}
\end{aligned}
$$

$$
\begin{aligned}
(\lambda x{:}T.t)t' &\;\to\; t[x := t'] \\
\pi f.l &\;\to\; f\,l \qquad \text{if } l \in \mathit{Dom}\, f
\end{aligned}
$$

$$\frac{}{T \leq \text{Top}} \qquad \frac{T_1' \leq T_1 \quad T_2 \leq T_2'}{T_1 \to T_2 \leq T_1' \to T_2'} \qquad \frac{\mathit{Dom}\,F' \subseteq \mathit{Dom}\,F \quad \forall l \in \mathit{Dom}\,F' \colon\; Fl \leq F'l}{\Pi F \leq \Pi F'}$$

$$\frac{\Gamma x = T}{\Gamma \mapsto x : T} \qquad \frac{\Gamma[x := T] \mapsto t : T'}{\Gamma \mapsto \lambda x{:}T.t : T \to T'} \qquad \frac{\Gamma \mapsto t : T'' \to T \quad \Gamma \mapsto t' : T' \quad T' \leq T''}{\Gamma \mapsto t\,t' : T}$$

$$\frac{\mathit{Dom}\,f = \mathit{Dom}\,F \quad \forall l \in \mathit{Dom}\,f \colon\; \Gamma \mapsto fl : Fl}{\Gamma \mapsto \pi f : \Pi F} \qquad \frac{\Gamma \mapsto t : \Pi F \quad Fl = T}{\Gamma \mapsto t.l : T}$$

$$\Gamma \vdash t : T \quad \overset{\text{def}}{\iff} \quad \exists T' \colon\; \Gamma \mapsto t : T' \;\wedge\; T' \leq T$$

Note that we have defined $\mapsto$ before $\vdash$. The following properties can be shown:

- $\leq$ is a partial order (by induction on types)
- $\mapsto$ has the Unique Type Property (by induction on types).
- $\vdash$ has the Least Type Property (obvious)
- $\leq$ and $\vdash$ have the Subsumption Property (by induction on terms).
- Preservation, Progress and Termination.

Proposition 2 (Lubs and Glbs)

1. Two types always have a least upper bound.
2. If two types have a lower bound, they have a greatest lower bound.