Beta reduction terminates on simply typed terms

We will show that the reduction relation of the simply typed lambda calculus terminates, even if every beta redex can be reduced. This result was first proved by William W. Tait and published in 1967. Our proof is adapted from Girard's book (1989, Chapter 6). The proof is a fascinating demonstration of inductive proof techniques.

Terms and types are given by

$$T \in Ty = X \mid T \to T$$
$$t \in Ter = x \mid \lambda x{:}T.t \mid t\,t$$

As always, we assume a de Bruijn term representation. To reduce bureaucracy, we assume a global type environment $\tau \in Var \to Ty$ such that for every type T there exist infinitly many variables x with $\tau x = T$. We use two notational conveniences:

$$\lambda x.t \overset{\text{def}}{=} \lambda x{:}\tau x.t$$
$$t : T \overset{\text{def}}{\iff} \tau \vdash t : T$$

We will use the following proposition:

Prop P

1. $x : T \iff \tau x = T$
2. $t_1 t_2 : T \iff \exists T'\colon\ t_1 : T' \to T \ \wedge\ t_2 : T'$
3. $\lambda x.t : T \iff \exists T'\colon\ T = \tau x \to T' \ \wedge\ t : T'$
4. $t \to t' \implies S\theta t \to S\theta t'$

A β**-redex** is a term of the form $(\lambda x.t)t'$. The set of **terminating terms** is defined recursively:

1. If t doesn't contain a β-redex, then t is terminating.
2. If $\forall t' : t \to t' \implies t'$ terminating, then t is terminating.

Some of the inductive proofs will be based on the terminating relation

$$t \to' t' \overset{\text{def}}{\iff} t \to t' \wedge t \text{ terminating}$$

Tait's proof proceeds as follows:

1. Define a set of admissible terms.
2. Show that every admissible term terminates.
3. Show that every (well-typed) term is admissible.

For each type T, we define a set A_T of admissible terms of type T. The sets A_T are defined by recursion on T.

$$A_X \stackrel{\text{def}}{=} \{\, t \mid t : X \text{ and } t \text{ terminates} \,\}$$
$$A_{T_1 \to T_2} \stackrel{\text{def}}{=} \{\, t \mid t : T_1 \to T_2 \text{ and } \forall t' \in A_{T_1} \colon tt' \in A_{T_2} \,\}$$

A term t of type T is admissible if $t \in A_T$.

Lemma A

1. $t \in A_T \implies t$ terminates
2. $t \in A_T \wedge t \to t' \implies t' \in A_T$
3. $t : T \;\wedge\; t$ no abstraction $\wedge\; (\forall t' \colon t \to t' \implies t' \in A_T) \implies t \in A_T$

Proof By induction on T, where all three claims are proven together, so that we get a strong induction hypothesis. Case analysis:

1. $T = X$. *Easy.*
2. $T = T_1 \to T_2$.
   - Proof of A1. *Easy.*
   - Proof of A2. *Easy.*
   - Proof of A3. Let t be a term that is not an abstraction and let
     $\forall t' \colon t \to t' \implies t' \in A_T$ (1)
     By the definition of A_T for functional T it suffices to show:
     $\forall t_1 \in A_{T_1} \colon tt_1 \in A_{T_2}$ (2)
     We show this by induction on t_1 with respect to $\to'$.
     Let $t_1 \in A_{T_1}$. (3)
     By the outer induction hypothesis it suffices to show:
     $\forall t_2 \colon tt_1 \to t_2 \implies t_2 \in A_{T_2}$.
     Let $tt_1 \to t_2$. Case analysis:
     a) $t_2 = t't_1$ and $t \to t'$. Then $t' \in A_T$ and $t_1 \in A_{T_1}$ by (1) and (3). Hence $t_2 = t't_1 \in A_{T_2}$.
     b) $t_2 = tt_1'$ and $t_1 \to t_1'$. Then $t_1 \to' t_1'$ and $t_1' \in A_{T_1}$ by (3) and (A1) and (A2) using the outer induction hypothesis. Hence $t_2 = tt_1' \in A_{T_2}$ by the inner induction hypothesis (see (2)). □

Note that (A3) implies that every variable is an admissible term. Hence there are admissible terms for every type.

Lemma B $\quad t : T \;\wedge\; (\forall t' \in A_{\tau x} \colon t[x := t'] \in A_T) \;\Longrightarrow\; \lambda x.t \in A_{\tau x \to T}$

Proof By induction on t with respect to $\to'$.
Let $\forall t' \in A_{\tau x} \colon t[x := t'] \in A_T$. (1)
By definition of reducibility it suffices to show:
$\forall t' \in A_{\tau x} \colon (\lambda x.t)t' \in A_T$. (2)
We show this by induction on t' with respect to $\to'$.
Let $t' \in A_{\tau x}$. (3)
By (A3) it suffices to show $\forall t'' \colon (\lambda x.t)t' \to t'' \Longrightarrow t'' \in A_T$. (4)
Let $(\lambda x.t)t' \to t''$. Case analysis:

1. $t'' = t[x := t']$. Then $t'' \in A_T$ by (1) and (3).
2. $t'' = (\lambda x.t)t'_1$ and $t' \to t'_1$. Then $t' \to' t'_1$ and $t'_1 \in A_{\tau x}$ by (3), (A1) and (A2).
   Hence $t'' \in A_T$ by the inner induction hypothesis (see (2)).
3. $t'' = (\lambda x.t_1)t'$ and $t \to t_1$. Then by (1) (with $t' = x$), (A3) and (A1):
   $t = t[x := x] \in A_T$ and $t \to' t_1$.
   By (3) it suffices to show: $\lambda x.t_1 \in A_{\tau x \to T}$.
   By the outer induction hypothesis it suffices to show:
   $\forall t_2 \in A_{\tau x} \colon t_1[x := t_2] \in A_T$.
   Let $t_2 \in A_{\tau x}$. Then $t[x := t_2] \in A_T$ by (1). Since $t \to t_1$, we have by (P4):
   $t[x := t_2] \to t_1[x := t_2]$.
   Hence $t_1[x := t_2] \in A_T$ by (A2). □

We will only consider substitutions $\theta \in \mathit{Var} \to \mathit{Ter}$ that map every variable x to a term of type τx.

Lemma C $\quad t : T \wedge (\forall x \colon \theta x \in A_{\tau x}) \;\Longrightarrow\; S\theta t \in A_T$

The proof is by induction on the size of t and uses (B).

Theorem $\quad \to$ terminating

Proof We show that all well-typed terms are terminating. Let $t : T$. Let θ be the identity substitution with $\theta x = x$ for all x. Then $\forall x \colon \theta x \in A_{\tau x}$ by (A3). Hence $t \in A_T$ by (C). Hence t is terminating by (A1). □