Assignment 4
Semantics, WS 2013/14

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Read in the lecture notes: Chapter 8 - 13

Abstraction, translation and weak reduction for SK-terms are defined in the lecture notes, pages 14-15. The Church numerals for CL are defined by:

$$\begin{aligned} zero &:= KI \\ succ &:= SB \\ \hat{n} &:= succ^n zero \end{aligned}$$

The reflexive, transitive closure of a relation R is written as R^*. The symmetric closure of a relation will be written $R^{\leftrightarrow}$.

Exercise 4.1 Prove $({}^x s)t \succ^* s^x_t$ in CL.

Exercise 4.2 Find two terms s and t such that $s \succ t$ but not $[s] \succ^*_w [t]$.
Hint: Consider $\lambda x.Ix$.

Exercise 4.3 Give a normal fixed point combinator for CL.
Hint: Translate Y.
Explain why the sample proof for Exercise 3.3 fails in CL.

Exercise 4.4 Prove $\hat{n} f x \ \succ^* \ f^n x$ in CL.

Exercise 4.5 Prove the following in lambda calculus using the definitions from CL.

$$\begin{aligned} zero &\succ^* \lambda f x.x \\ succ &\succ^* \lambda n f x.\ f(n f x) \\ \hat{n} &\succ^* \lambda f x.\ f^n x \end{aligned}$$

Exercise 4.6 Compute the normal forms of the following terms in CBV-lambda calculus.

a) $II(II)$
b) $K(\omega(\lambda xy.\Omega))$

Exercise 4.7 Let $\equiv_V$ be the least equivalence relation containing call-by-value reduction. Give terms s, t_1, t_2 such that $t_1 \equiv_V t_2$, but $s\, t_1 \not\equiv_V s\, t_2$.

Exercise 4.8 (Properties of *star*, Coq) Prove some properties for the inductive definition of the reflexive transitive closure of a relation R.

a) Soundness: Show that R^* is reflexive, transitive and contains R.
b) Monotonicity: If $R \subseteq S$, then $R^* \subseteq S^*$.
c) Completeness: Show that if S is any reflexive, transitive relation containing R, then $R^* \subseteq S$.
d) Idempotence: Show that $(R^*)^* = R^*$.

Exercise 4.9 (Properties of *con*, Coq) Prove some properties for the inductive definition of the relfexive transitive symmetric closure of a relation R.

a) Show that $\mathit{con}\,R$ is transitive and symmetric.

b) Prove $\mathit{con}\,R = (R^{\leftrightarrow})^*$

Exercise 4.10 (Alternative axiomatizations for equivalence, Coq, optional) A relation R is an equivalence relation if it is reflexive, symmetric and transitive. This is one of many possible definitions. In particular it is not even the most concise definition.

Show that every total relation which is (left) Euclidean is an equivalence relation. A relation is (left) Euclidean if from $R\ x\ z$ and $R\ y\ z$ we can infer that $R\ x\ y$. The intuition behind this definition is attributed to Euclid, who demands that "things which equal the same thing also equal one another" in his axiomatization of equivalence.

Exercise 4.11 (Coq, optional) In the lecture we discussed a possible alternative definition for the reflexive, transitive, symmetric closure of a relation. This definition does not work and the point of this exercise is to analyze it more precisely. In the coq development, we define *bcon* to be the bad definition for conversion. Show the following:

$$(R^*)^{\leftrightarrow} \subset \mathit{bcon} \subset (R^{\leftrightarrow})^*$$

In particular, both inclusions are strict.