Confluence and Normalization in Reduction Systems Lecture Notes

Gert Smolka
Saarland University

December 16, 2015

We study confluence and normalization in abstract reduction systems and apply the results to combinatory logic and an abstract version of the lambda beta calculus. For both systems we obtain the Church-Rosser property and a complete normalization strategy. For an abstract weak call-by-value lambda calculus we obtain uniform confluence, which ensures that for a given term all maximal reduction chains have the same length. The development is based on constructive type theory with inductive predicates and is realized in Coq.

1 Introduction

The Church-Rosser theorem formulates a key property of lambda-calculus and other rewriting systems. It turns out that the Church-Rosser property can be analysed in the abstract by considering relations on an abstract type. One speaks of abstract reduction systems since one-step reduction turns out to be the primary relation.

Abstract reduction systems can be seen as directed graphs, a view providing for strong intuitions and graphical proof sketches.

The study of reduction systems profits much from inductively defined predicates, with strong normalization and (parallel) reduction as prominent examples.

2 Relations

We assume a type X and call predicates $X \to X \to \mathsf{Prop}$ **relations**. The letters R and S will range over relations. Inclusion and equivalence of relations are defined as follows:

$$R \subseteq S \;:=\; \forall xy.\, Rxy \to Sxy$$
$$R \cong S \;:=\; R \subseteq S \land S \subseteq R$$

Reflexivity, symmetry, transitivity, and functionality of relations are defined as follows:

$$\begin{aligned}
\text{reflexive } R \;&:=\; \forall x.\, Rxx\\
\text{symmetric } R \;&:=\; \forall xy.\, Rxy \to Ryx\\
\text{transitive } R \;&:=\; \forall xyz.\, Rxy \to Ryz \to Rxz\\
\text{functional } R \;&:=\; \forall xyz.\, Rxy \to Rxz \to y = z
\end{aligned}$$

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive. We define the **inverse** and the **symmetric closure** of relations:

$$\begin{aligned}
R^{-} \;&:=\; \lambda xy.\, Ryx\\
R^{\leftrightarrow} \;&:=\; \lambda xy.\, Rxy \lor Ryx
\end{aligned}$$

Fact 1 Let R be reflexive. Then R^{-} and $R^{\leftrightarrow}$ are reflexive.

Fact 2 Let R be transitive. Then R^{-} and $R^{\leftrightarrow}$ are transtive.

Composition, **powers**, and **union** of relations are defined as follows:

$$\begin{aligned}
R \circ S \;&:=\; \lambda xz.\, \exists y.\, Rxy \land Syz\\
R^{0} \;&:=\; \lambda xy.\, x{=}y\\
R^{n+1} \;&:=\; R \circ R^{n}\\
R \cup S \;&:=\; \lambda xy.\, Rxy \lor Sxy
\end{aligned}$$

Fact 3 The following statements are equivalent: R is symmetric; R^{-} is symmetric; $R^{-} \subseteq R$; $R \cong R^{-}$; $R \cong R^{\leftrightarrow}$.

Fact 4

a) R reflexive iff $R^{0} \subseteq R$.

b) R transitive iff $R \circ R \subseteq R$.

c) $R^{\leftrightarrow} \cong R \cup R^{-}$.

d) $R^{m+n} \cong R^{m} \circ R^{n}$.

We say that R is a **least relation** satisfying a property p if pR and $R \subseteq S$ for every relation S with pS. Note that least relations satisfying a property are unique up to equivalence.

Exercise 5 Prove $R^{m+n} \cong R^{m} \circ R^{n}$.

Exercise 6 Prove that R^{n} is functional if R is functional.

3 Reflexive Transitive Closure

We define **reflexive transitive closure** of a relation R as an inductive predicate $R^*: X \to X \to \mathsf{Prop}$:

$$\frac{}{R^*yy} \qquad\qquad \frac{Rxx' \qquad R^*x'y}{R^*xy}$$

Fact 7 (Expansion) $R \subseteq R^*$.

Fact 8 (Transitivity) R^* is transitive.

Proof Let R^*xy and R^*yz. We show R^*xz by induction on R^*xy.

1. Let $x = y$. Then the assumption R^*yz is the claim.
2. Let Rxx' and $R^*x'y$. We have $R^*x'z$ by the inductive hypothesis. The claim follows with the second rule defining star. ∎

Fact 9 (Monotonicity) If $R \subseteq S$, then $R^* \subseteq S^*$.

Proof Let $R \subseteq S$ and R^*xy. We show S^*xy by induction R^*xy.

1. Let $x = y$. The claim is S^*xx and follows by the first rule defining star.
2. Let Rxx' and $R^*x'y$. The Sxx' by the assumption and $S^*x'y$ by the inductive hypothesis. The claim follows with the second rule defining star. ∎

We write R^{**} for $(R^*)^*$.

Fact 10 $R^{**} \subseteq R^*$.

Proof Let $R^{**}xy$. We show R^*xy by induction on the outer star of $R^{**}xy$.

1. Let $x = y$. The claim is R^*xx and follows by the first rule defining star.
2. Let R^*xx' and $R^{**}x'y$. We have $R^*x'y$ by the inductive hypothesis. The claim follows by transitivity of R^*. ∎

Fact 11 (Idempotence) $R^{**} \cong R^*$.

Proof Follows with Facts 7 and 10. ∎

We formulate the induction principle for R^* explicitly.

Fact 12 (Star induction) Let R^*xy. Then px if py and $\forall ab.\, Rab \to pb \to pa$.

Proof Let py and $\forall ab.\, Rab \to pb \to pa$. We prove px by induction on R^*xy.

1. Let $x = y$. The claim follows with the first assumption.
2. Let Rxx' and $R^*x'y$. We have px' by the inductive hypothesis. The claim follows with the second assumption. ■

Fact 13 (Closure) R^* is a least reflexive and transitive relation containing R.

Fact 14 (Power characterization) $R^*xy \leftrightarrow \exists n. R^n xy$.

Fact 15 $R^{-*} \cong R^{*-}$.

Fact 16 $R^{*\leftrightarrow} \subseteq R^{\leftrightarrow *}$.

Exercise 17 Prove all facts stated above in Coq. Coq generates and uses a suboptimal induction lemma for R^*xy that doesn't treat y as a parameter. Thus the inductive hypotheses come with an unnecessary premise.

Discussion

We have defined R^* as an inductive predicate. In the literature, the reflexive transitive closure is usually defined using the power characterization stated by Fact 14 (e.g., [5, 1]). The power characterization relies on recursion and induction for numbers and requires no knowledge of inductive predicates. We are using an inductive definition because we like its simplicity. It captures the notion of reflexive transitive closure without the notion of numbers. Although Coq generates a suboptimal induction lemma for the inductive definition of R^*, the inductive definition is still more convenient to use than a power-based definition. The reader may use the examples of this section to refamiliarise himself with inductive definitions and their realization in Coq.

Exercise 18 Find a counterexample for $R^{*\leftrightarrow} \cong R^{\leftrightarrow *}$.

Exercise 19 Prove $(R^* \cup S^*)^* \cong (R \cup S)^*$ using monotonicity and idempotence of star. No induction is needed.

Exercise 20 Given R, define an inductive predicate $\hat{R} : \mathbf{N} \to X \to X \to \mathsf{Prop}$:

$$\frac{}{\hat{R}nxx} \qquad\qquad \frac{Rxy \qquad \hat{R}nyz}{\hat{R}(n+1)xz}$$

Prove $\hat{R}n \cong R^n$. Remark: In Coq, working with $\hat{R}$ rather than R^n provides for more direct inductive proofs.

Exercise 21 Given R, define an inductive predicate $R^\# : X \to X \to \mathsf{Prop}$:

$$\frac{Rxy}{R^\# xy} \qquad \frac{}{R^\# xx} \qquad \frac{R^\# xy \quad R^\# yz}{R^\# xz}$$

Prove $R^\# \cong R^*$.

Exercise 22 (Impredicative Characterization)
Prove $R^*xy \leftrightarrow \forall S.\, S^0 \subseteq S \to R \circ S \subseteq S \to Sxy$.
Note that the equivalence characterises R^* as the intersection of a all reflexive relations that are closed under left-composition with R.

4 Equivalence Closure

Fact 23 (Preservation of Symmetry) If R is symmetric, then R^* is symmetric.

Proof Let R be symmetric and R^*xy. We show R^*yx by induction on R^*xy.

1. Let $x = y$. The claim is R^*xx and follows by the assumption.
2. Let Rxx' and $R^*x'y$. We have $R^*x'x$ by the assumption and expansion, and R^*yx' by the inductive hypothesis. The claim follows by transitivity. ∎

We define $R^+ := R \circ R^*$ and $R^\equiv := R^{\leftrightarrow *}$. We call R^+ the **transitive closure** and $R^\equiv$ the **equivalence closure** of R.

Fact 24 $R \subseteq R^+ \subseteq R^* \subseteq R^\equiv$.

Fact 25 (Transitive Closure) R^+ is a least transitive relation containing R.

Fact 26 (Equivalence Closure) $R^\equiv$ is a least equivalence relation containing R.

5 Confluence and Church-Rosser Property

We define **joinability**, **confluence**, **semi-confluence**, the **diamond property**, and the **Church-Rosser property** as follows:

$$\begin{aligned}
\text{joinable } Rxy &:= \exists z.\, Rxz \wedge Ryz \\
\text{diamond } R &:= \forall xyz.\, Rxy \to Rxz \to \text{joinable } Ryz \\
\text{confluent } R &:= \text{diamond } R^* \\
\text{semi-confluent } R &:= \forall xyz.\, Rxy \to R^*xz \to \text{joinable } R^*yz \\
\text{Church-Rosser } R &:= \forall xy.\, R^\equiv xy \to \text{joinable } R^*xy
\end{aligned}$$

Fact 27 The following statements are equivalent:

1. R is Church-Rosser.
2. R is confluent.
3. R is semi-confluent.

Proof The directions $1 \Longrightarrow 2 \Longrightarrow 3$ are straightforward. We show $3 \Longrightarrow 1$.

Let R be semi-confluent and $R^{\leftrightarrow *}xy$. We show joinable R^*xy by induction on $R^{\leftrightarrow *}xy$. If $x = y$, the claim is trivial. Otherwise, let $R^{\leftrightarrow}xx'$ and $R^{\leftrightarrow *}x'y$. We have $R^*x'z$ and R^*yz for some z by the inductive hypothesis. Case analysis on $R^{\leftrightarrow}xx'$. If Rxx', the claim follows. Otherwise, we have $Rx'x$. By semi-confluence of R we obtain some u such that R^*xu and R^*zu. Thus R^*yu by transitivity. The claim follows. ■

Fact 28 If R satisfies the diamond property, then R is semi-confluent.

Proof Assume R satisfies the diamond property and let R^*xy and Rxz. We show joinable R^*yz by induction on R^*xy. If $x = y$, the claim is trivial. Otherwise, let Rxx' and $R^*x'y$. By the diamond property, we have $Rx'u$ and Rzu for some u. By the inductive hypothesis for $R^*x'y$, we have joinable R^*yu. The claim follows. ■

Fact 29 (Sandwich) Let $R \subseteq S \subseteq R^*$. Then:

1. R is confluent iff S is confluent.
2. R is confluent if S satisfies the diamond property.

Proof We have $R^* \subseteq S^* \subseteq R^*$ by monotonicity and idempotence of star. Thus $R^* \cong S^*$. Thus Claim 1 follows. Claim 2 follows with Facts 28 and 27. ■

A relation R is **strongly confluent** if

$$\forall xyz.\, Rxy \to Rxz \to (R^*yz \vee \exists u.\, R^*yu \wedge Rzu)$$

Fact 30 Every relation satisfying the diamond property is strongly confluent.

Fact 31 Every strongly confluent relation is confluent.

Proof By Fact 27 it suffices to show that a strongly confluent relation is semi-confluent. This follows with an induction similar to the one used for Fact 28. ■

Exercise 32 Show that star preserves the diamond property. That is, if R satisfies the diamond property, then R^* satisfies the diamond property.

Exercise 33 Prove that the following statements are equivalent:

1. diamond R.
2. $\forall xyzn.\, Rxy \to R^n xz \to \exists u.\, R^n yu \wedge Rzu$.
3. $\forall xyzmn.\, R^m xy \to R^n xz \to \exists u.\, R^n yu \wedge R^m zu$.

Exercise 34 Prove that the following statements are equivalent:

1. $\forall xyz.\, Rxy \to Sxz \to \exists u.\, Syu \wedge Rzu$.
2. $\forall xyzn.\, Rxy \to S^n xz \to \exists u.\, S^n yu \wedge Rzu$.
3. $\forall xyzmn.\, R^m xy \to S^n xz \to \exists u.\, S^n yu \wedge R^m zu$.

Exercise 35 Prove that a relation R is Church-Rosser if and only if $R^{\equiv} \cong R^* \circ R^{*-}$.

6 Normal Forms

We define **reducibility**, **normality**, and **normal forms**:

reducible Rx	$:=$	$\exists y.\, Rxy$	**x is R-reducible**
normal Rx	$:=$	$\neg$reducible Rx	**x is R-normal**
$R^{\Downarrow}xy$	$:=$	$R^*xy \wedge$ normal Ry	**y is R-normal form of x**

Fact 36 Let x be R-normal. Then:

1. If R^*xy, then $x = y$.
2. If $R^n xy$, then $n = 0$ and $x = y$.

Fact 37 Let R be confluent. Then:

1. $R^{\Downarrow}$ is functional. That is, no x has more than one R-normal form.
2. $R^{\Downarrow}xy$ iff $R^{\equiv}xy$ and y is R-normal.
3. If x and y are R-normal, then $x = y$ iff $R^{\equiv}xy$.
4. If x and y are R-normal, then $\neg R^{\equiv}xy$ iff $x \neq y$.

We say that R is **classical** if every x is either R-reducible or R-normal.

7 Triangle Method

We now introduce the triangle method, which provides for elegant confluence proofs for lambda calculus and combinatory logic. The triangle method also provides a reduction strategy that finds a normal form whenever there is one.

We start with a definition. A function $\rho : X \to X$ is a **triangle operator for R** if $Ry(\rho x)$ whenever Rxy.

Fact 38 Let ρ be a triangle operator for R. Then:
1. R satisfies the diamond property.
2. If x is reducible, then $R^2x(\rho x)$.
3. If R is reflexive, then $Rx(\rho x)$.

Fact 39 Let ρ be a triangle operator for R. Then:
1. If Rxy, then $R(\rho x)(\rho y)$.
2. If Rxy, then $R(\rho^n x)(\rho^n y)$.
3. If R^*xy, then $R^*y(\rho^n x)$ for some n.

A function $\rho : X \to X$ is a **normalizer for R** if it satisfies the following properties:
1. $R^*x(\rho x)$.
2. If $x \Downarrow^R y$, then $\rho^n x = y$ for some n.

Fact 40 Let ρ be a normalizer for R. Then y is an R-normal form of x if and only if y is R-normal and $y = \rho^n x$ for some n.

Theorem 41 (Triangle) Let $R \subseteq S \subseteq R^*$, S be reflexive, and ρ be a triangle operator for S. Then R is confluent and ρ is a normalizer for R.

Proof Confluence of R follows with Facts 29 and 38. We have $R^* \cong S^*$ by the assumption $R \subseteq S \subseteq R^*$ and monotonicity and idempotence of $*$. That ρ is a normalizer for R now follows with the reflexivity of S and Fact 39. ∎

Exercise 42 We call x **R-quasi-normal** if $x = y$ whenever Rxy. We call y an **R-quasi normal form of x** if R^*xy and y is quasi-normal in R. Prove the following claims:

a) Let y and z be R-quasi normal forms of x. Then $y = z$ if R is confluent.
b) If y is an R-normal form of x, then y is an R^*-quasi-normal form of x.
c) Let ρ be a triangle operator for R and y be an R-quasi-normal form of x. Then $\rho y = y$ and $y = \rho^n x$ for some n.

8 Confluence of Combinatory Logic SK

Figure 1 shows the definition of the combinatory logic SK. We show that the reduction relation $\succ$ is confluent and give a normalizer for $\succ$. The proof is based on the triangle method and sandwiches a relation known as parallel reduction. The definitions of parallel reduction $\gg$ and the triangle operator ρ appear in Figure 2. A term is called a **redex** if it has the form Ks or $Sstu$.

$$s, t ::= x|K|S|st \qquad (x : \mathbf{N})$$

$$\frac{}{Kst \succ s} \qquad \frac{}{Sstu \succ su(tu)} \qquad \frac{s \succ s'}{st \succ s't} \qquad \frac{t \succ t'}{st \succ st'}$$

Figure 1: Definition of SK

$$\frac{s \gg s'}{Kst \gg s'} \qquad \frac{s \gg s' \qquad t \gg t' \qquad u \gg u'}{Sstu \gg s'u'(t'u')} \qquad \frac{}{s \gg s} \qquad \frac{s \gg s' \qquad t \gg t'}{st \gg s't'}$$

$$\begin{aligned}
\rho(Kst) &:= \rho s \\
\rho(Sstu) &:= (\rho s)(\rho u)((\rho t)(\rho u)) \\
\rho(st) &:= (\rho s)(\rho t) && \text{if } st \text{ not a redex} \\
\rho(s) &:= s && \text{if } s \text{ not an application}
\end{aligned}$$

Figure 2: Parallel reduction and triangle operator for SK

Informally, we may understand a parallel reduction $s \gg t$ as a two phase process: First one chooses a collection of redexes in s and then one reduces these redexes one after the other following an innermost strategy. For this to work it is essential that a redex is not destroyed if its constituents are reduced. This is clearly the case of SK.

The notion of parallel reduction is elegantly captured by its inductive definition in Figure 2. Defining parallel reduction formally following the informal explanation and not using an inductive definition is not an easy exercise. The inductive definition of parallel reduction and its use for confluence proofs is due to Tait and Martin-Löf.

Fact 43 (Triangle) ρ is a triangle operator for $\gg$.

Proof Let $s \gg t$. We prove $t \gg \rho s$ by induction on s following the case analysis in the definition of ρ. If s is not an application, then $s = t$ and the claim is trivial. If s is an application but not a redex, then $s \gg t$ is obtained with a non-redex rule for $\gg$. Hence the claim is either trivial or follows with the inductive hypothesis. We consider the case $s = Ks_1s_2$. The case $s = Ss_1s_2s_3$ is similar.

Let $s = Ks_1s_2$. We show $t \gg \rho s_1$. If t is obtained with the rule for K, then $s_1 \gg t$ and the claim follows with the inductive hypothesis. If t is obtained with the reflexivity rule, we have $t = Ks_1s_2$ and the claim follows with the inductive

$$s, t ::= n \mid st \mid \lambda s \qquad (n : \mathbf{N})$$

$$\frac{}{(\lambda s)t \succ \beta s t} \qquad \frac{s \succ s'}{st \succ s't} \qquad \frac{t \succ t'}{st \succ st'} \qquad \frac{s \succ s'}{\lambda s \succ \lambda s'}$$

Figure 3: Definition of abstract $\lambda\beta$

hypothesis. If t is obtained with the compatibility rule for applications, we have $t = Ks_1's_2'$ and $s_1 \gg s_1'$ and $s_2 \gg s_2'$. The claim follows with the inductive hypothesis. ▪

Fact 44 (Compatibility) Let $s \succ^* s'$ and $t \succ^* t'$. Then $st \succ^* s't'$.

Proof By induction on $t \succ^* t'$ with nested induction on $s \succ^* s'$. ▪

Fact 45 (Sandwich) $\succ \subseteq \gg \subseteq \succ^*$.

Proof The first inclusion follows by induction on $\succ$. The second inclusion follows by induction on $\gg$ using compatibility (Fact 44). ▪

Theorem 46 (SK) $\succ$ is confluent and ρ is a normalizer for $\succ$.

Proof Follows with Theorem 41 and Facts 43 and 45. ▪

9 Confluence of Abstract $\lambda\beta$

Figure 3 defines an abstract version of the lambda beta calculus we call **abstract** $\lambda\beta$. The definition assumes a function β that given two terms yields a term. We will need one assumption about β to show confluence of abstract $\lambda\beta$. The proof will be based on the triangle method and will exhibit a normalizer for abstract $\lambda\beta$. The definitions of parallel reduction $\gg$ and the triangle operator ρ appear in Figure 4.

Fact 47 (Compatibility) Let $s \succ^* s'$. Then $\lambda s \succ^* \lambda s'$.

Proof By induction on $s \succ^* s'$. ▪

Fact 48 (Compatibility) Let $s \succ^* s'$ and $t \succ^* t'$. Then $st \succ^* s't'$.

Proof By induction on $t \succ^* t'$ with nested induction on $s \succ^* s'$. ▪

Fact 49 (Reflexivity) $s \gg s$.

Proof By induction on s. ▪

$$\frac{s \gg s' \qquad t \gg t'}{(\lambda s)t \gg \beta s't'} \qquad \frac{}{x \gg x} \qquad \frac{s \gg s' \qquad t \gg t'}{st \gg s't'} \qquad \frac{s \gg s'}{\lambda s \gg \lambda s'}$$

$$\begin{array}{rcll} \rho((\lambda s)t) & := & \beta(\rho s)(\rho t) & \\ \rho(st) & := & (\rho s)(\rho t) & \text{if } s \text{ not an abstraction} \\ \rho(\lambda s) & := & \lambda(\rho s) & \\ \rho(x) & := & x & \end{array}$$

Figure 4: Parallel reduction and triangle operator for abstract $\lambda\beta$

Fact 50 (Sandwich) $\succ \subseteq \gg \subseteq \succ^*$.

Proof The first inclusion follows by induction on $\succ$ using Fact 49. The second inclusion follows by induction on $\gg$ using compatibility (Facts 48 and 47). ∎

We say that β is **compatible** if $\beta st \gg \beta s't'$ whenever $s \gg s'$ and $t \gg t'$.

Fact 51 (Triangle) Let β be compatible. Then ρ is a triangle operator for $\gg$.

Proof Let $s \gg t$. We prove $t \gg \rho s$ by induction on $s \gg t$. The case for the β-rule follows with the compatibility of β. No other case needs the compatibility assumption. The cases for variables and abstractions are straightforward. The final case is for the compatibility rule for applications. If $s = xs'$ or $s = s_1s_2s_3$, the claim follows with the inductive hypotheses. Otherwise, $s = (\lambda s_1)s_2$, $t = (\lambda s_1')s_2'$, $s_1 \gg s_1'$, and $s_2 \gg s_2'$. We need to show $(\lambda s_1')s_2' \gg \beta(\rho s_1)(\rho s_2)$, which follows with the induction hypotheses $s_1' \gg \rho s_1$ and $s_2' \gg \rho s_2$. ∎

Theorem 52 (Abstract $\lambda\beta$) Let β be compatible. Then $\succ$ is confluent and ρ is a normalizer for $\succ$.

Proof Follows with Theorem 41 and Facts 51 and 50. ∎

Exercise 53 Give an example that shows that parallel reduction is not transitive.

10 Equivalence in Abstract $\lambda\beta$

Recall the definition $R^{\equiv} := R^{\leftrightarrow *}$. We now show that $\succ^{\equiv}$ for abstract $\lambda\beta$ agrees with an inductive predicate $\equiv$ defined as follows:

$$\frac{}{(\lambda s)t \equiv \beta st} \qquad \frac{s \equiv s' \qquad t \equiv t'}{st \equiv s't'} \qquad \frac{s \equiv s'}{\lambda s \equiv \lambda s'} \qquad \frac{}{s \equiv s} \qquad \frac{s \equiv t}{t \equiv s} \qquad \frac{s \equiv t \qquad t \equiv u}{s \equiv u}$$

We can see $s \equiv t$ as the least compatible equivalence relation on terms satisfying the abstract β-law.

Fact 54 (Compatibility)

1. If $s \succ^{\equiv} s'$ and $t \succ^{\equiv} t'$, then $st \succ^{\equiv} s't'$.
2. If $s \succ^{\equiv} s'$, then $\lambda s \succ^{\equiv} \lambda s'$.

Proof Both claims follow by star induction. The proofs are similar to the proofs of Facts 48 and 47. ∎

Fact 55 $\succ \subseteq \equiv$.

Proof By induction on $\succ$. ∎

Fact 56 (Coincidence) $\succ^{\equiv} \cong \equiv$.

Proof Let $s \succ^{\leftrightarrow *} t$. We prove $s \equiv t$ by star induction on $s \succ^{\leftrightarrow *} t$. If $s = t$, the claim is trivial. Otherwise, we have $s \succ^{\leftrightarrow} s'$ and $s' \equiv t$ by the inductive hypothesis. By Fact 55 we have either $s \equiv s'$ or $s' \equiv s$. The claim follows.

Let $s \equiv t$. We prove $s \succ^{\leftrightarrow *} t$ by induction on $s \equiv t$. The cases for the β-rule and the reflexivity rule are straightforward. The case for the transitivity rule follows with Fact 8, and the case for the symmetry rule follows with Fact 23. The cases for the compatibility rules follow with Fact 54. ∎

11 Uniform Confluence

We define **uniform confluence** of relations as follows:

$$\text{uniformly confluent } R \;:=\; \forall xyz.\, Rxy \to Rxz \to y{=}z \lor \exists u.\, Ryu \land Rzu$$

Fact 57 Every functional relation is uniformly confluent.

Fact 58 Every relation satisfying the diamond property is uniformly confluent.

Fact 59 Every uniformly confluent relation is strongly confluent.

Fact 60 The following statements are equivalent:

1. R uniformly confluent.
2. $\forall xyzn.\, Rxy \to R^n xz \to \exists ukl.\, R^k yu \land R^l zu \land 1 + k = n + l \land k \le n \land l \le 1$.
3. $\forall xyzmn.\, R^m xy \to R^n xz \to \exists ukl.\, R^k yu \land R^l zu \land m + k = n + l \land k \le n \land l \le m$.

Proof We prove $1 \implies 2 \implies 3 \implies 1$. The implication $3 \implies 1$ is straightforward.

$$s, t \ ::= \ n \mid st \mid \lambda s \qquad (n : \mathbf{N})$$

$$\frac{}{(\lambda s)(\lambda t) \succ \beta s(\lambda t)} \qquad \frac{s \succ s'}{st \succ s't} \qquad \frac{t \succ t'}{st \succ st'}$$

Figure 5: Definition of abstract L

1. $1 \Longrightarrow 2$. Assume (1) and Rxy and $R^n xz$. We prove by induction on n that $R^k yu$ and $R^l zu$ for some u, $k \leq n$ and $l \leq 1$ such that $1 + k = n + l$. If $n = 0$, then $x = z$ and the claim follows with $u = y$ and $l = 1$. Otherwise, we have Rxx' and $R^{n-1}x'z$. Case analysis using (1).
   a) $y = x'$. The claim follows with $u = z$, $k = n - 1$, and $l = 0$.
   b) Rxv and $Rx'v$ for some v. By the inductive hypothesis, we obtain u, $k' \leq n - 1$ and $l' \leq 1$ such that $R^{k'}vu$, $R^l zu$, and $1 + k' = n - 1 + l$. The claim follows with $k = 1 + k'$.
2. $2 \Longrightarrow 3$. Assume (2) and $R^m xy$ and $R^n xz$. We prove by induction on m that $R^k yu$ and $R^l zu$ for some u, $k \leq n$ and $l \leq 1$ such that $m + k = n + l$. If $m = 0$, then $x = y$ and the claim follows with $u = z$, $k = n$, and $l = 0$. Otherwise, we have Rxx' and $R^{m-1}x'y$. By (2) we obtain v, $l' \leq 1$, and $k' \leq n$ such that $R^{k'}x'v$, $R^{l'}zv$, and $1 + k' = n + l'$. By the inductive hypothesis for $R^{m-1}x'y$ we obtain u, $k \leq k'$, and $l'' \leq m - 1$ such that $R^k yu$, $R^{l''}vu$, and $m - 1 + k = k' + l''$. The claim follows with $l = l' + l''$. ∎

Fact 61 (Uniform normalization) Let R be uniformly confluent, $R^m xy$, $R^n xz$, and z be R-normal. Then $m \leq n$ and $R^{n-m}yz$.

Proof Follows with Facts 60 and 36. ∎

12 Uniform Confluence of Abstract L

Figure 5 defines an abstract version of the weak call-by-value λ-calculus we call **abstract L**. We show that the reduction relation $\succ$ is uniformly confluent. This result holds without any assumption on the function β. Intuitively, uniform confluence holds for L since in L can only reduce outermost redexes.

Fact 62 (Uniform confluence) Let $s \succ t_1$ and $s \succ t_2$. Then either $t_1 = t_2$ or $t_1 \succ u$ and $t_2 \succ u$ for some u.

Proof By induction on $s \succ t_1$.

1. Let $s = (\lambda s_1)(\lambda s_2)$. Then $t_1 = t_2$ since abstractions are irreducible.
2. Let $s = s_1 s_2$, $s_1 \succ s_1'$, and $t_1 = s_1' s_2$. Case analysis on $s \succ t_2$.
   a) $s_1 \succ s_1''$ and $t_2 = s_1'' s_2$. The claim follows with the inductive hypothesis for $s_1 \succ s_1'$.
   b) $s_2 \succ s_2'$, and $t_2 = s_1 s_2'$. The claim follows with $u = s_1' s_2'$.
3. Analogous to (2). ∎

Exercise 63 Define a normalizer for one-step reduction in abstract L.

13 Strong Normalization

We define two inductive predicates **WN** Rx (**weakly normalizing**) and **SN** Rx (**strongly normalizing**):

$$\frac{\mathsf{normal}\ Rx}{\mathsf{WN}\ Rx} \qquad \frac{Rxy \qquad \mathsf{WN}\ Ry}{\mathsf{WN}\ Rx} \qquad \frac{\forall y.\ Rxy \to \mathsf{SN}\ Ry}{\mathsf{SN}\ Rx}$$

We say that R is **terminating** if SN Rx for every x.

Fact 64 SN Rx if x is R-normal.

Fact 65 (Unfolding) SN $Rx \leftrightarrow \forall y.\ Rxy \to$ SN Ry.

Fact 66 (SN induction) Let SN Rx. Then px if
$\forall a.\ \mathsf{SN}\ Ra \to (\forall b.\ Rab \to pb) \to pa$.

Fact 67 (Well-founded induction) Let R be terminating. Then px if
$\forall a.\ (\forall b.\ Rab \to pb) \to pa$.

Fact 68 WN Rx iff x has an R-normal form.

Fact 69 Let R be classical and SN Rx. Then WN Rx.

Proof By induction on SN Rx. ∎

Fact 70 Let SN Rx and R^*xy. Then SN Ry.

Proof By induction on R^*xy. ∎

Fact 71 Let SN Rx. Then SN R^+x.

Proof By induction on SN Rx. By unfolding of SN R^+x we assume R^+xy and prove SN R^+y. We have Rxx' and $R^*x'y$ for some x', and SN R^+x' by the inductive hypothesis. Case analysis on $R^*x'y$.

1. $x' = y$. The claim is trivial since we have SN R^+x'.
2. $R^+x'y$. The claim follows by unfolding of SN R^+x'. ∎

Fact 72 (Homomorphism) Let R be a relation on X and S be a relation on A. Let $f : X \to A$ be a function such that $S(fx)(fy)$ whenever Rxy. Then SN Rx if SN $S(fx)$.

Proof Let $p := \lambda a.\forall x.\, fx{=}a \to \mathsf{SN}\ Rx$. We prove SN $Sa \to pa$ for all a by induction on SN Sa. We assume IH : $\forall b.\, Sab \to pb$ and prove pa. We assume $fx = a$ and prove SN Rx. By unfolding, we assume Rxy and prove SN Ry. We have $Sa(fy)$ by the assumption. By IH, we have $p(fy)$. The claim SN Ry follows. ∎

Fact 73 (Uniform confluence)
Let R be uniformly confluent and WN Rx. Then SN Rx.

Proof We have R^nxy and normal Ry by Fact 68. We prove SN Rx by induction on n. For $n = 0$, we have $x = y$ and the claim follows by Fact 64. Otherwise, we have Rxx' and $R^{n-1}x'y$ for some x'. By unfolding of the claim, we assume Rxx'' and prove SN Rx''. By the inductive hypothesis, it suffices to show that $R^{n-1}x''y$, which follows by uniform normalization (Fact 61). ∎

Exercise 74 Give a relation on $\{0, 1\}$ such that 0 is weakly normalizing but not strongly normalizing.

Exercise 75 Let SN Rx. Prove $\neg Rxx$.

Exercise 76 (Anti-monotonicity) Let $R \subseteq S$ and SN Sx. Prove SN Rx using Fact 72.

Exercise 77 Show SN Rx if SN R^+x.

14 Divergence

The dual notion for strong normalization is divergence. A point *diverges* if there in an infinite path of R-steps issuing from the point. If we assume excluded middle, every point is either strongly normalizing or diverging.

We define divergence with **progressive sets**. A **set** is a unary predicate on X. A set p is **progressive** if for every x in p there is some y in p such that Rxy. A point **diverges** if there exists a progressing set containing it. Formally:

$$\begin{aligned} \mathsf{progressive}\ Rp &:= \forall x.\, px \to \exists y.\, Rxy \land py && \textbf{progressive set} \\ \mathsf{diverges}\ Rx &:= \exists p.\, px \land \mathsf{progressive}\ Rp && \textbf{diverging point} \end{aligned}$$

Fact 78 (Disjointness) There is no x such that both $\mathsf{SN}\ Rx$ and $\mathsf{diverges}\ Rx$.

Proof We assume $\mathsf{SN}\ Rx$ and show $\neg\mathsf{diverges}\ Rx$. By induction on $\mathsf{SN}\ Rx$, we assume $\mathsf{IH} : \forall a.\ Rxa \to \neg\mathsf{diverges}\ Ra$ and prove $\neg\mathsf{diverges}\ Rx$. We assume $\mathsf{diverges}\ Rx$ and prove falsity. The assumption gives us a progressive set p such that Rxy and y is in p. Thus Rxy and y diverges, which contradicts IH. ∎

Fact 79 (Exhaustiveness) Assume excluded middle. Then, for every x, either $\mathsf{SN}\ Rx$ or $\mathsf{diverges}\ Rx$.

Proof Assume $\neg\mathsf{SN}\ Rx$. By excluded middle, it suffices to show that x diverges. Let $p := \lambda z.\ \neg\mathsf{SN}\ Rz$. It suffices to show that p is progressive. We assume pa und show that there is some b such that Rab and pb. We have $\neg\forall b.\ Rab \to \mathsf{SN}\ Rb$ by unfolding. The claim follows by excluded middle. ∎

Exercise 80 Let $x : X$. Show that $\forall R.\ \mathsf{SN}\ Rx \vee \mathsf{diverges}\ Rx$ implies excluded middle.

15 Local Confluence

We define **local confluence** of relations as follows:

$$\mathsf{locally\ confluent}\ R \ :=\ \forall xyz.\ Rxy \to Rxz \to \exists u.\ R^*yu \wedge R^*zu$$

Example 81 There are locally confluent relations that are not confluent. For instance, $R12$, $R21$, $R10$ and $R23$.

Fact 82 (Newman's Lemma) Let R be terminating and locally confluent. Then R is confluent.

Proof Let $px := \forall yz.\ R^*xy \to R^*xz \to \exists u.\ R^*yu \wedge R^*zu$. It suffices to prove px for all x. By well-founded induction we assume $\mathsf{IH} : \forall b.\ Rxb \to pb$ and prove px. By the definition of p we assume R^*xy and R^*xz and prove that R^*yu and R^*zu for some u. If $x = y$ or $x = z$, the claim is trivial. Otherwise we have Rxx_1, R^*x_1y, Rxx_2, and R^*x_2z. By local confluence we have R^*x_1v and R^*x_2v for some v. By IH we have some w such that R^*vw and R^*zw. By transitivity of R^* and IH we have R^*yu and R^*wu for some u. The claim follows by transitivity of R^*. ∎

16 Historical Remarks

The study of abstract reduction systems originated with Newman [7]. The notions of confluence, semi-confluence, diamond property, uniform confluence, and local confluence all appear in Newman [7] (see Theorems 1–3). The modern, relational view of abstract reduction systems is due to Huet [5]. Baader and Nipkow's textbook [1] on term rewriting starts with a chapter on abstract reduction systems.

Newman's lemma appears as Theorem 3 in [7]. The constructive proof based on well-founded induction is due to Huet [5]. Newman's original proof is not constructive and does not employ well-founded induction. Newman's lemma was one of the first proofs done with Coq [2].

Newman's lemma is an example of a result where the constructive proof is shorter and clearer than the classical proof. Newman defined termination as the absence of infinite path. That well-founded induction is valid for relations disallowing infinite paths was first observed by Emmy Noether. This fact can only be shown with excluded middle.

The inductive definition of strong normalization seems to originate with Coquand and Huet's [2] definition of noetherian relations and Huet's [5] use of well-founded induction (noetherian induction) for the proof of Newman's lemma. Newman [7], Huet [5], and Baader and Nipkow [1] define strong normalization as the absence of infinite paths, thus foregoing a constructive proof of well-founded induction.

The notion of parallel reduction goes back to Curry and Feys [3]. The inductive definition of parallel reduction and its use for confluence proofs is due to Tait (1965) and Martin-Löf (1971) (see Hindley and Seldin [4], Appendix A2). Triangle operators originated with Takahashi's [10] confluence proof for $\lambda\beta$.

The name uniform confluence is from [9, 8]. Niehren [8] observes that the weak call-by-value λ-calculus is uniformly confluent. Dal Lago and Martini [6] prove Fact 60 for the weak call-by-value λ-calculus.

There is much interesting material not covered in these notes, including strong normalization proofs for typed lambda calculi and confluence and termination techniques for first order rewriting systems [1].

References

[1] Franz Baader and Tobias Nipkow. *Term rewriting and all that.* Cambridge University Press, 1998.

[2] Thierry Coquand and Gérard Huet. Constructions: A higher order proof system for mechanizing mathematics. In *Proc. of EUROCAL'85, Linz, Austria*, volume 203 of *LNCS*, pages 151–184. Springer, 1985.

[3] Haskell B. Curry and Robert Feys. *Combinatory Logic, Volume I.* North-Holland Publishing Company, 1958.

[4] J. Roger Hindley and Jonathan P. Seldin. *Lambda-Calculus and Combinators, an Introduction*. Cambridge University Press, 2008.

[5] Gérard Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *Journal of the ACM (JACM)*, 27(4):797–821, 1980.

[6] Ugo Dal Lago and Simone Martini. The weak lambda calculus as a reasonable machine. *Theor. Comput. Sci.*, 398(1-3):32–50, 2008.

[7] Maxwell Herman Alexander Newman. On theories with a combinatorial definition of "equivalence". *Annals of Mathematics*, 43(2):223–243, 1942.

[8] Joachim Niehren. Uniform confluence in concurrent computation. *J. Funct. Program.*, 10(5):453–499, 2000.

[9] Joachim Niehren and Gert Smolka. A confluent relational calculus for higher-order programming with constraints. In Jean-Pierre Jouannaud, editor, *Constraints in Computational Logics, First International Conference, Munich, Germany, September 7-9, 1994*, volume 845 of *LNCS*, pages 89–104. Springer, 1994.

[10] Masako Takahashi. Parallel reductions in λ-calculus. *Inf. Comput.*, 118(1):120–127, 1995.