Confluence

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We study confluence of abstract relations and an abstract $\lambda\beta$-calculus parameterized with a function for β-reduction. We show confluence of the abstract $\lambda\beta$-calculus using parallel reduction and a Takahashi function. We also study evaluation, strong normalization, and uniform confluence.

1 Relations

Given a type X, we call predicates $X \to X \to \mathbf{P}$ **relations**. The letters R and S will range over relations. Inclusion and equivalence of relations are defined as follows:

$$R \subseteq S \;:=\; \forall xy.\, Rxy \to Sxy$$
$$R \cong S \;:=\; R \subseteq S \land S \subseteq R$$

Reflexivity, symmetry, transitivity, and functionality of relations are defined as follows:

$$\text{reflexive } R \;:=\; \forall x.\, Rxx$$
$$\text{symmetric } R \;:=\; \forall xy.\, Rxy \to Ryx$$
$$\text{transitive } R \;:=\; \forall xyz.\, Rxy \to Ryz \to Rxz$$
$$\text{functional } R \;:=\; \forall xyz.\, Rxy \to Rxz \to y = z$$

2 Reflexive Transitive Closure

Let $R : X \to X \to \mathbf{P}$. We define the **reflexive transitive closure** R^* of R as an inductive predicate:

$$\frac{}{R^*xx} \qquad\qquad \frac{Rxx' \qquad R^*x'y}{R^*xy}$$

We refer to the induction lemma for R^* as **star induction**. Moreover, we refer to the constructor mapping R to R^* as **star**.

Fact 1

1. R is reflexive and transitive.
2. *Expansion* $R \subseteq R^*$.
3. *Monotonicity* If $R \subseteq S$, then $R^* \subseteq S^*$.
4. *Completeness* If $R \subseteq S$ and S is reflexive and transitive, then $R^* \subseteq S$.
5. *Idempotence* $R^{**} \cong R^*$.

Proof Transitivity, monotonicity, and completeness follow with star induction. Idempotence is a straightforward consequence of expansion, completeness, and transitivity. ■

Note that Fact 1 tells us that R^* is the least reflexive and transitive relation containing R.

Exercise 2 There are several equivalent definitions of the reflexive transitive closure of a relation. Consider the inductive predicate $R^\#$ defined by the following rules:

$$\frac{Rxy}{R^\# xy} \qquad \frac{}{R^\# xx} \qquad \frac{R^\# xy \quad R^\# yz}{R^\# xz}$$

Prove $R^\# \cong R^*$.

Exercise 3 Define composition $R \circ S$ and powers R^n of relations and prove $R^{n+1} \cong R \circ R^n \cong R^n \circ R$ and $R^* xy \leftrightarrow \exists n.\, R^n xy$.

3 Basic Confluence

Let $R : X \to X \to \mathbf{P}$. We define:

$$\begin{aligned} \text{joinable } Rxy &:= \exists z.\, Rxz \land Ryz \\ \text{diamond } R &:= \forall xyz.\, Rxy \to Rxz \to \text{joinable } Ryz \\ \text{confluent } R &:= \text{diamond } R^* \\ \text{semi-confluent } R &:= \forall xyz.\, Rxy \to R^*xz \to \text{joinable } R^*yz \end{aligned}$$

Fact 4 (Diamond) R is semi-confluent if R satisfies the diamond property.

Proof Let R satisfy the diamond property. Let Rxy_1 and R^*xy_2 We show by induction on R^*xy_2 that y_1 and y_2 are joinable. If $x = y_2$, the claim is trivial. Otherwise, we have Rxx' and $R^*x'y_2$. By the diamond property, we have Ry_1u and $Rx'u$ for some u. By the inductive hypothesis for $R^*x'y_2$, we have joinable R^*uy_2. The claim follows. ■

Note that Fact 4 says that star preserves the diamond property (i.e., R^* satisfies the diamond property if R satisfies the diamond property).

Fact 5 (Semi-Confluence) R is confluent iff R is semi-confluent.

Proof The direction from confluence to semi-confluence is obvious. For the other direction, let R be semi-confluent, R^*xy_1 and R^*xy_2. We show by induction on R^*xy_1 that y_1 and y_2 are joinable. If $x = y_1$, the claim is trivial. Otherwise, let Rxx' and $R^*x'y_1$. By semi-confluence of R, we have $R^*x'u$ and R^*y_2u for some u. By the inductive hypothesis for $R^*x'y_1$, we have joinable R^*y_1u. The claim follows. ■

We now define the prediamond property:

$$\text{prediamond } R \;:=\; \forall xyz.\, Rxy \to Rxz \to y{=}z \lor y \succ z \lor z \succ y \lor \text{joinable } R\ y\ z$$

The prediamond property is weaker than the diamond property but still implies semi-confluence.

Fact 6 (Prediamond)

1. If R satisfies the diamond property, then R satisfies the prediamond property.
2. If R satisfies the prediamond property, then R is confluent.

Proof Claim 1 is trivial. For Claim 2 it suffices by Fact 5 to show that R is semi-confluent. This follows with a straightforward adaption of the proof of Fact 4. ■

Example 7 It does not seem possible to further relax the prediamond property without loosing confluence. For instance, $R12$, $R21$, $R10$ and $R23$ is a relation that is not confluent.

4 Evaluation and Normal Forms

Let $\succ : X \to X \to \mathbf{P}$. We define **reducible** and **normal** points as follows:

$$\begin{aligned} \text{reducible } x &:= \exists y.\, x \succ y \\ \text{normal } x &:= \neg\text{reducible } x \end{aligned}$$

Normal points may also be called *irreducible* or *terminal* points.

The **evaluation relation** for $\succ$ is defined as follows:

$$x \rhd y \;:=\; x \succ^* y \land \text{normal } y$$

If $x \rhd y$, we say that x **evaluates** to y or that y is a **normal form** of x. Moreover, we say that x is **weakly normalizing** if it has a normal for.

We now show that confluence ensures uniqueness of normal forms.

Fact 8 If $x \succ^* y$ and x is normal, then $x = y$.

Proof Case analysis on $x \succ^* y$. ∎

Fact 9 If $\succ$ is confluent, then $\rhd$ is functional.

Proof Follows with Fact 8. ∎

A **reduction function for** $\succ$ is a function $\rho : X \to X$ such that $x \succ^* \rho x$ and $x \rhd y \to \exists n.\ \rho^n x = y$ for all x and y.

Fact 10 Let ρ be a reduction function. Then $\rho^n x = x$ if x is normal.

Proof Follows with Fact 8. ∎

Let normality for $\succ$ be decidable. We define a function

$$E : (X \to X) \to \mathsf{N} \to X \to \mathsf{O}X$$

satisfying the equations

$$\begin{aligned} E\rho 0x &= \emptyset \\ E\rho(Sn)x &= \mathsf{if\ normal}\ x\ \mathsf{then}\ \lfloor x \rfloor\ \mathsf{else}\ E\rho n(\rho x) \end{aligned}$$

Fact 11 Let ρ be a reduction function for $\succ$. Then:

1. If $\rho^n x$ normal, then $E\rho(Sn)x = \lfloor \rho^n x \rfloor$.
2. if $E\rho nx = \lfloor y \rfloor$, then $x \rhd y$.
3. $x \rhd y \leftrightarrow \exists n.\ E\rho nx = \lfloor y \rfloor$.

Proof Claim 1 follows by induction on n using Fact 10. Claim 2 follows by induction on n. Claim 3 follows from claims 1 and 2. ∎

Exercise 12 Let ρ be a reduction function. Show that $x \rhd y$ if and only if y is normal and $\rho^n x = y$ for some n.

5 Strong Normalisation

Informally, termination may be defined as the absence of infinite paths. This characterization doesn't say much constructively. However, there is an elegant inductive definition of termination that provides a strong induction lemma and thus works constructively.

We define an inductive predicate SN Rx:

$$\frac{\forall y.\, Rxy \to \mathsf{SN}\, Ry}{\mathsf{SN}\, Rx}$$

We say that x is **strongly normalizing** in R if SN Rx. Moreover, we say that R is **terminating** if SN Rx for every x.

Fact 13 SN Rx if x is R-normal.

Fact 14 (Unfolding) SN $Rx \leftrightarrow \forall y.\, Rxy \to$ SN Ry.

Fact 15 (SN induction)
$(\forall x.\, \mathsf{SN}\, Rx \to (\forall y.\, Rxy \to py) \to px) \to (\forall x.\, \mathsf{SN}\, Rx \to px)$.

As one can see from the proposition formulating SN induction, the use of SN induction is very natural since it simply adds the inductive hypothesis for all successors to the proof goal. Incidentally, the induction lemma Coq generates for SN replaces the premise SN Rx with the equivalent premise $\forall y.\, Rxy \to$ SN Ry.

Fact 16 Let SN Rx and R^*xy. Then SN Ry.

Proof By induction on R^*xy. ■

Fact 17 (Morphism) Let R be a relation on X and S be a relation on A. Let $f : X \to A$ be a function such that $S(fx)(fy)$ whenever Rxy. Then SN Rx if SN $S(fx)$.

Proof Let $p := \lambda a. \forall x.\, fx{=}a \to$ SN Rx. We prove SN $Sa \to pa$ for all a by induction on SN Sa. We assume IH : $\forall b.\, Sab \to pb$ and prove pa. We assume $fx = a$ and prove SN Rx. By unfolding, we assume Rxy and prove SN Ry. We have $Sa(fy)$ by the assumption. By IH, we have $p(fy)$. The claim SN Ry follows. ■

The morphism lemma is very useful in practice. For instance, if one wants to show that SN(st) implies SN s in a λ-calculus, one can simply apply the morphism lemma with the morphism $fu := ut$.

Constructively, one cannot show in general that a strongly normalizing point has a normal form. A relation is **classical** if reducibility is logically decidable (i.e., a point is either reducible or irreducible).

Fact 18 In a classical relation, strongly normalizing points are weakly normalizing.

Proof By SN induction. ■

Fact 19 (Transitive closure) Let $R^+xy := \exists x'.\, Rxx' \wedge R^*x'y$.
Then SN $Rx \leftrightarrow$ SN R^+x.

Proof Both directions follow by SN induction. We show the direction from left to right, the other direction is routine.

Let SN Rx. We show SN R^+x by induction on SN Rx. By unfolding of SN R^+x we assume R^+xy and prove SN R^+y. Case analysis.

1. Rxy. Thus SN R^+y by the inductive hypothesis.
2. Rxx' and $R^+x'y$ for some x'. The SN R^+x' by the inductive hypothesis. Thus SN R^+y by unfolding. ∎

Exercise 20 Give a relation on $\{0, 1\}$ such that 0 has a normal form but is not strongly normalizing.

Exercise 21 Let SN Rx. Prove $\neg Rxx$.

Exercise 22 Let $R \subseteq S$ and SN Sx. Prove SN Rx.

Exercise 23 A common but limited technique for proving that a relation R is terminating is to give a function f such that $Rxy \to fx > fy$ for all x and y.

a) Prove that the relation $m > n$ on N is terminating.
b) Prove that a relation R is terminating if there is a function $f : X \to \mathsf{N}$ such that $Rxy \to fx > fy$ for all x and y. Use the morphism lemma.
c) Consider the following inductively defined relation R on $\mathsf{O}(\mathsf{N})$:

$$\frac{}{R\,\lfloor Sn\rfloor\,\lfloor n\rfloor} \qquad\qquad \frac{}{R\,\emptyset\,\lfloor n\rfloor}$$

   i) Draw R as a graph.
   ii) Prove that R is terminating.
   iii) Prove that there is no function $f : \mathsf{O}(\mathsf{N}) \to \mathsf{N}$ such that $Rxy \to fx > fy$.

6 Newman's Lemma

We define **local confluence** of relations as follows:

$$\text{locally confluent } R \;:=\; \forall xyz.\, Rxy \to Rxz \to \text{joinable } R^*\,y\,z$$

Clearly, relations satisfying the prediamond property are locally confluent. We may ask whether locally confluent relations are always confluent. It turns out that there are finite locally confluent relations that are not confluent. Example 7 provides such a counterexample. On the positive side, we can show that every terminating relation is confluent if it is locally confluent. This fact is known as Newman's lemma. There is an elegant proof of Newman's lemma using SN induction.

Fact 24 (Well-founded induction) Let R be terminating. Then px if $\forall x.\ (\forall y.\ Rxy \to py) \to px$.

Proof Follows with Fact 15. ∎

Fact 25 (Newman's Lemma)
Let R be terminating and locally confluent. Then R is confluent.

Proof Let $px := \forall yz.\ R^*xy \to R^*xz \to \exists u.\ R^*yu \land R^*zu$. It suffices to prove px for all x. By well-founded induction we have the inductive hypothesis $\forall y.\ Rxy \to py$.

By definition of p we assume R^*xy and R^*xz and prove that y and z are joinable in R^*. If $x = y$ or $x = z$, the claim is trivial. Otherwise we have Rxx_1, R^*x_1y, Rxx_2, and R^*x_2z. We will use the inductive hypothesis for both x_1 and x_2.

By local confluence we have R^*x_1u and R^*x_2u for some u. By the inductive hypothesis for x_2 we have some v such that R^*uv and R^*zv. By the inductive hypothesis for x_1 and transitivity of R^* we have R^*yw and R^*vw for some w. Joinability of y and z now follows by transitivity of R^*. ∎

7 Uniform Confluence

We define **uniform confluence** of relations as follows:

$$\text{uniformly\_confluent } R \;:=\; \forall xyz.\ Rxy \to Rxz \to y{=}z \lor \text{joinable } R\ y\ z$$

Fact 26

1. Every functional relation is uniformly confluent.
2. Every relation satisfying the diamond property is uniformly confluent.
3. Every uniformly confluent relation satisfies the prediamond property and thus is confluent.

We will show that for a uniformly confluent relation all reductions of a given point to a normal form have the same length.

We define **graded reduction** R^nxy as follows:

$$\frac{}{R^0xx} \qquad\qquad \frac{Rxx' \qquad R^nx'y}{R^{Sn}xy}$$

Fact 27

1. $R^n \subseteq R^*$ and $R^1 \cong R$.
2. If R^*xy, then R^nxy for some n.

3. If $R^m xy$ and $R^n yz$, then $R^{m+n} xz$.
4. If $R^n xy$ and x is normal in R, then $n = 0$ and $x = y$.

We define **graded joinability** as follows:

$$\text{graded\_joinable } Ryzmn \;:=\; \exists ukl.\, R^k yu \wedge R^l zu \wedge m + k = n + l \wedge k \leq n \wedge l \leq m$$

Fact 28 The following statements are equivalent:

1. R uniformly confluent.
2. $\forall xyzn.\, Rxy \to R^n xz \to \text{graded\_joinable } Ryz1n$.
3. $\forall xyzmn.\, R^m xy \to R^n xz \to \text{graded\_joinable } Ryzmn$.

Proof We prove $1 \to 2 \to 3 \to 1$. The implication $3 \to 1$ is straightforward.

- $1 \to 2$. The proof refines the proof of Fact 4. Assume (1) and Rxy and $R^n xz$. We prove by induction on n that $R^k yu$ and $R^l zu$ for some u, $k \leq n$ and $l \leq 1$ such that $1 + k = n + l$. If $n = 0$, then $x = z$ and the claim follows with $u = y$ and $l = 1$. Otherwise, we have Rxx' and $R^{n-1} x'z$. Case analysis using (1).
  - $y = x'$. The claim follows with $u = z$, $k = n - 1$, and $l = 0$.
  - Rxv and $Rx'v$ for some v. By the inductive hypothesis, we obtain u, $k' \leq n - 1$ and $l' \leq 1$ such that $R^{k'} vu$, $R^l zu$, and $1 + k' = n - 1 + l$. The claim follows with $k = 1 + k'$.
- $2 \to 3$. The proof refines the proof of Fact 5, direction semi-confluence to confluence. Assume (2) and $R^m xy$ and $R^n xz$. We prove byinduction on m that $R^k yu$ and $R^l zu$ for some u, $k \leq n$ and $l \leq 1$ such that $m + k = n + l$. If $m = 0$, then $x = y$ and the claim follows with $u = z$, $k = n$, and $l = 0$. Otherwise, we have Rxx' and $R^{m-1} x'y$. By (2) we obtain v, $l' \leq 1$, and $k' \leq n$ such that $R^{k'} x'v$, $R^{l'} zv$, and $1 + k' = n + l'$. By the inductive hypothesis for $R^{m-1} x'y$ we obtain u, $k \leq k'$, and $l'' \leq m - 1$ such that $R^k yu$, $R^{l''} vu$, and $m - 1 + k = k' + l''$. The claim follows with $l = l' + l''$. ∎

Fact 29 (Uniform normalization) Let R be uniformly confluent, $R^m xy$, $R^n xz$, and z be normal for R. Then $m \leq n$ and $R^{n-m} yz$.

Proof Follows with Facts 28 and 27. ∎

Fact 30 Let R be uniformly confluent. Then every point that has a normal form is strongly normalizing.

Proof Let $R^* xy$ and y be normal. Then $R^n xy$ for some n by Fact 27. We prove SN Rx by induction on n. For $n = 0$, we have $x = y$ and the claim follows by Fact 13. Otherwise, we have Rxx' and $R^{n-1} x'y$ for some x'. By unfolding of the claim, we assume Rxx'' and prove SN Rx''. By the inductive hypothesis, it suffices to show $R^{n-1} x''y$, which follows by uniform normalization (Fact 29). ∎

Fact 31 Let $\succ\colon X \to X \to \mathsf{P}$ be uniformly confluent and $\rho : X \to X$ satisfy $x \succ^* \rho x$ and $\rho x = x \to \mathsf{normal}\ x$ for x. Then ρ is a reduction function for $\succ$.

Exercise 32 Give a confluent relation $\succ$ and a function ρ satisfying the conditions of Fact 31 that is not a reduction function for $\succ$.

8 TMT Method

The confluence of λ-calculi where reduction is possible within abstractions can be shown with a clever method building on work of Tait, Martin-Löf, and Takahashi. We speak of the *TMT method*. The TMT method factorises in three part:

1. An abstract part not making assumptions about terms. The abstract part yields confluence and a reduction function.
2. An intermediate part using terms but keeping substitution abstract.
3. A concrete part dealing with the concrete substitution used.

We present the abstract part in the following. The key idea is to show the confluence of a relation R by identifying a suitable auxiliary relation S such that $R \subseteq S \subseteq R^*$ and S has the diamond property.

Fact 33 (Sandwich) Let $R \subseteq S \subseteq R^*$. Then:

1. $R^* \cong S^*$.
2. If S has the diamond property, then R is confluent.

Proof Claim 1 follows with monotonicity and idempotence of star. Claim 2 follows with (1) and the fact that star preserves the diamond property. ∎

A **Takahashi function for $\succ$** is a function $\rho : X \to X$ such that $x \succ y \to y \succ \rho x$ for all x and y.

Fact 34 (Takahashi) Let ρ be a Takahashi function for $\succ$. Then:

1. *Diamond* If $x \succ y_1$ and $x \succ y_2$, then $y_1 \succ \rho x$ and $y_1 \succ \rho x$.
2. *Soundness* If $\succ$ is reflexive, then $x \succ \rho x$.
3. *Preservation* If $x \succ y$, then $\rho x \succ \rho y$.
4. *Cofinality* If $x \succ^* y$, then $y \succ^* \rho^n x$ for some n.

Proof Claim 1: Immediate from the Takahashi property of ρ.

Claim 2: Given x, we have $x \succ x$ by reflexivity. Thus $x \succ \rho x$.

Claim 3: Let $x \succ y$. Then $y \succ \rho x \succ \rho y$.

Claim 4: Let $x \succ^* y$. We prove $\exists n.\ y \succ^* \rho^n x$ by induction on $x \succ^* y$. The first subgoal is trivial. In the second subgoal we have $x \succ x' \succ^* y \succ^* \rho^n x'$ for

$$s,t ::= n \mid st \mid \lambda s \qquad (n : \mathbf{N})$$

$$\frac{}{(\lambda s)t \succ \beta st} \qquad \frac{s \succ s'}{st \succ s't} \qquad \frac{t \succ t'}{st \succ st'} \qquad \frac{s \succ s'}{\lambda s \succ \lambda s'}$$

Figure 1: Definition of abstract $\lambda\beta$

$$\frac{s \gg s' \qquad t \gg t'}{(\lambda s)t \gg \beta s't'} \qquad \frac{}{x \gg x} \qquad \frac{s \gg s' \qquad t \gg t'}{st \gg s't'} \qquad \frac{s \gg s'}{\lambda s \gg \lambda s'}$$

$$\begin{aligned} \rho((\lambda s)t) &= \beta(\rho s)(\rho t) & \\ \rho(st) &= (\rho s)(\rho t) & \text{if } s \text{ not an abstraction} \\ \rho(\lambda s) &= \lambda(\rho s) & \\ \rho(x) &= x & \end{aligned}$$

Figure 2: Parallel reduction and Takahashi function for abstract $\lambda\beta$

some n using the inductive hypothesis. By Claim 3 we have $\rho^n x \succ \rho^n x'$. Thus $\rho^n x' \succ \rho(\rho^n x)$. Hence $y \succ^* \rho^{Sn} x$. ■

Theorem 35 (TMT) Let $\succ$ and $\gg$ be predicates $X \to X \to \mathbf{P}$ such that $\succ \subseteq \gg \subseteq \succ^*$ and $\gg$ is reflexive. Moreover, let ρ be a Takahashi function for $\gg$. Then $\succ$ is confluent and ρ is a reduction function for $\succ$.

Proof Follows with Facts 33, 34, and 8. ■

9 Confluence of Abstract Lambda Beta

Figure 1 defines an abstract version of the lambda beta calculus we call **abstract** $\lambda\beta$. The definition assumes a function β that given two terms yields a term. We will need one assumption about β to show confluence of abstract $\lambda\beta$ using the TMT method.

The key idea is the definition of an intermediate relation $\succ \subseteq \gg \subseteq \succ^*$ known as **parallel reduction**. The definitions of parallel reduction $\gg$ and the accompanying Takahashi function ρ appear in Figure 2.

Fact 36 (Compatibility)

1. If $s \succ^* s'$ and $t \succ^* t'$, then $st \succ^* s't'$.
2. If $s \succ^* s'$, then $\lambda s \succ^* \lambda s'$.

Proof By star induction. For Claim 1, one starts with induction on $s \succ^* s'$ and continues with a nested induction on $t \succ^* t'$ in the base case. ■

Fact 37 (Reflexivity) $s \gg s$.

Proof By induction on s. ■

Fact 38 (Sandwich) $\succ \subseteq \gg \subseteq \succ^*$.

Proof The first inclusion follows by induction on $\succ$ using Fact 37. The second inclusion follows by induction on $\gg$ using compatibility (Fact 36). ■

We say that β is **compatible** if $\beta st \gg \beta s't'$ whenever $s \gg s'$ and $t \gg t'$.

Fact 39 Let β be compatible. Then ρ is a Takahashi function for $\gg$.

Proof Let $s \gg t$. We prove $t \gg \rho s$ by induction on $s \gg t$. The case for the β-rule follows with the compatibility of β. No other case needs the compatibility assumption. The cases for variables and abstractions are straightforward. The final case is for the compatibility rule for applications. If $s = xs'$ or $s = s_1 s_2 s_3$, the claim follows with the inductive hypotheses. Otherwise, $s = (\lambda s_1)s_2$, $t = (\lambda s_1')s_2'$, $s_1 \gg s_1'$, and $s_2 \gg s_2'$. We need to show $(\lambda s_1')s_2' \gg \beta(\rho s_1)(\rho s_2)$, which follows with the induction hypotheses $s_1' \gg \rho s_1$ and $s_2' \gg \rho s_2$. ■

Theorem 40 Let β be compatible. Then $\succ$ is confluent and ρ is a reduction function for $\succ$.

Proof Follows with Theorem 35 and Facts 39 and 38. ■

Exercise 41 Give an example that shows that parallel reduction is not transitive.

Exercise 42 Define an abstract call-by-value lambda calculus and show that it is uniformly confluent. In contrast to abstract $\lambda\beta$, no condition for β is needed.

Exercise 43 (Confluence of SK-Calculus) The SK-calculus employs applicative terms

$$s, t \ ::= \ x \mid K \mid S \mid st \qquad (x : \mathsf{N})$$

obtained with variables, two constants K and S, and application. Reduction in the SK-calculus is defined as follows:

$$\frac{}{Kst \succ s} \qquad \frac{}{Sstu \succ su(tu)} \qquad \frac{s \succ s'}{st \succ s't} \qquad \frac{t \succ t'}{st \succ st'}$$

Prove that reduction in the SK-calculus is confluent. Use the TMT method with a suitably defined parallel reduction.

Some background. SK is also known as combinatory logic and may be seen as a subsystem of $\lambda\beta$ where $K = \lambda xy.x$ and $S = \lambda fgx.(fx)(gx)$. SK can express all computable functions and has been extensively studied by logicians.

10 Equivalence Closure

General λ-calculi are best understood as deductive systems where one or several reduction rules generate an equivalence relation $s \equiv t$ on terms. If the reduction relation $s \succ t$ generated by the rules is confluent, as is typically the case for λ-calculi, there is a beautiful and useful connection between equivalence (deduction) and reduction (computation) known as Church-Rosser property:

$$s \equiv t \;\to\; \exists u.\, s \succ^* u \wedge t \succ^* u$$

From the Church-Rosser property one obtains the rule

$$\text{normal } t \to s \equiv t \to s \rhd t$$

which makes it possible to verify a computational claim $s \rhd t$ by showing the equivalence $s \equiv t$ by means of undirected equational reasoning. This is exploited for the verification of programming techniques for λ-calculus like Church numerals and Curry's fixed point combinator.

We first study the connection between reduction and equivalence in the abstract and then apply the results to abstract $\lambda\beta$.

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

We define the **symmetric closure** $R^{\leftrightarrow}$ of a relation R as follows:

$$R^{\leftrightarrow} \;:=\; \lambda xy.\, Rxy \vee Ryx$$

We write $R^{\leftrightarrow *}$ for $(R^{\leftrightarrow})^*$ and call $R^{\leftrightarrow *}$ the **equivalence closure of R**.

Fact 44

1. $R \subseteq R^{\leftrightarrow}$ and $R^{\leftrightarrow}$ is symmetric.
2. $R^* \subseteq R^{\leftrightarrow *}$.

3. If R is symmetric, then R^* is symmetric.
4. $R^{\leftrightarrow *}$ is an equivalence relation.
5. If $R \subseteq S$ and S is an equivalence relation, then $R^{\leftrightarrow *} \subseteq S$.

Note that Fact 44 tells us that $R^{\leftrightarrow *}$ is the least equivalence relation containing R. We define the **Church-Rosser property** for relations as follows:

$$\text{Church-Rosser } R \;:=\; \forall xy.\, R^{\leftrightarrow *}xy \to \text{joinable } R^* xy$$

Fact 45 R is Church-Rosser if and only if R is confluent.

Proof The direction from Church-Rosser to confluence is obvious since $R^* \subseteq R^{\leftrightarrow *}$ (monotonicity of star).

For the other direction we assume that R is semi-confluent and that $R^{\leftrightarrow *}xy$. We show joinable $R^* xy$ by star induction on $R^{\leftrightarrow *}xy$. If $x = y$, the claim is trivial. Otherwise, let $R^{\leftrightarrow}xx'$ and $R^{\leftrightarrow *}x'y$. We have $R^* x'z$ and $R^* yz$ for some z by the inductive hypothesis. Case analysis on $R^{\leftrightarrow}xx'$. If Rxx', the claim follows. Otherwise, we have $Rx'x$. By semi-confluence of R we obtain some u such that $R^* xu$ and $R^* zu$. Thus $R^* yu$ by transitivity. The claim follows. ∎

Fact 46 Let R be confluent, $R^{\leftrightarrow *}xy$, and y be normal for R. Then $x \rhd_R y$.

Proof Follows with Facts 45 and 8. ∎

Equivalence for abstract $\lambda\beta$ is defined as follows:

$$\frac{}{(\lambda s)t \equiv \beta st} \qquad \frac{s \equiv s' \quad t \equiv t'}{st \equiv s't'} \qquad \frac{s \equiv s'}{\lambda s \equiv \lambda s'} \qquad \frac{}{s \equiv s} \qquad \frac{s \equiv t}{t \equiv s} \qquad \frac{s \equiv t \quad t \equiv u}{s \equiv u}$$

We may say that $s \equiv t$ is the least compatible equivalence relation on terms satisfying the abstract β-law. It is now routine to prove that $s \equiv t$ agrees with $s \succ^{\leftrightarrow *} t$, where $\succ$ is the reduction relation for $\lambda\beta$ defined in Figure 1.

Lemma 47

1. If $s \succ^{\leftrightarrow *} s'$ and $t \succ^{\leftrightarrow *} t'$, then $st \succ^{\leftrightarrow *} s't'$.
2. If $s \succ^{\leftrightarrow *} s'$, then $\lambda s \succ^{\leftrightarrow *} \lambda s'$.
3. If $s \succ t$, then $s \equiv t$.

Proof Claims 1 and 2 follow with star induction, Claim 3 follows with induction on $\succ$. ∎

Fact 48 (Coincidence) $s \equiv t \;\leftrightarrow\; s \succ^{\leftrightarrow *} t$.

Proof Let $s \equiv t$. We prove $s \succ^{\leftrightarrow *} t$ by induction on $s \equiv t$. The case for the β-rule is straightforward. The cases for the compatibility rules follow with Lemma 47. The case for the equivalence rules follow with Fact 44 (4).

Let $s \succ^{\leftrightarrow *} t$. We prove $s \equiv t$ by star induction on $s \succ^{\leftrightarrow *} t$. If $s = t$, the claim is trivial. Otherwise, we have $s \succ^{\leftrightarrow} s'$ and $s' \equiv t$ by the inductive hypothesis. By Lemma 47 (3) we have either $s \equiv s'$ or $s' \equiv s$. The claim follows. ∎

Exercise 49 There are several equivalent definitions of the equivalence closure of a relation. Consider the inductive predicate $\equiv_R$ defined by the following rules:

$$\frac{Rxy}{x \equiv_R y} \qquad \frac{}{x \equiv_R x} \qquad \frac{x \equiv_R y}{y \equiv_R x} \qquad \frac{x \equiv_R y \quad y \equiv_R z}{x \equiv_R z}$$

Prove $\equiv_R \cong R^{\leftrightarrow *}$.

11 Historical Remarks

The study of abstract reduction systems originated with Newman [7]. The notions of confluence, semi-confluence, diamond property, and uniform confluence all appear in Newman [7] (see Theorems 1–3). The modern, relational view of abstract reduction systems is due to Huet [5]. Baader and Nipkow's textbook [1] on term rewriting starts with a chapter on abstract reduction systems.

Newman's lemma appears as Theorem 3 in [7]. The constructive proof based on well-founded induction is due to Huet [5]. Newman's original proof is not constructive and does not employ well-founded induction. Newman's lemma was one of the first proofs done with Coq [2].

Newman's lemma is an example of a result where the constructive proof is shorter and clearer than the classical proof. Newman defined termination as the absence of infinite path. That well-founded induction is valid for relations disallowing infinite paths was first observed by Emmy Noether. This fact can only be shown with excluded middle.

The inductive definition of strong normalization seems to originate with Coquand and Huet's [2] definition of noetherian relations and Huet's [5] use of well-founded induction (noetherian induction) for the proof of Newman's lemma. Newman [7], Huet [5], and Baader and Nipkow [1] define strong normalization as the absence of infinite paths, thus foregoing a constructive proof of well-founded induction.

The notion of parallel reduction goes back to Curry and Feys [3]. The inductive definition of parallel reduction and its use for confluence proofs is due to Tait (1965) and Martin-Löf (1971) (see Hindley and Seldin [4], Appendix A2). Takahashi functions originated with Takahashi's [10] confluence proof for $\lambda\beta$.

The name uniform confluence is from [9, 8]. Niehren [8] observes that the call-by-value λ-calculus is uniformly confluent. Dal Lago and Martini [6] prove Fact 28 for the call-by-value λ-calculus.

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