Equivalence of System F and $\lambda 2$: A Case Study of Context Morphisms

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Contributions

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A: Strong equivalence of two variants of System F [Girard '72, Reynolds '74]

- F with explicit, separate context for type variables, e.g. [Harper '13]
- $\lambda 2$, a pure type system (PTS) [Barendregt '91]
- Notion of Equivalence: *reduction of type checking* in both directions.

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A: Strong equivalence of two variants of System F [Girard '72, Reynolds '74]

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- Notion of Equivalence: *reduction of type checking* in both directions.

B: Methodology & Best Practices

- Take syntax with binders seriously - else it will bite you!
- Pursue a *big-step approach*:
  - de Bruijn syntax with parallel substitutions [Schäfer et al. CPP'15/ITP'15]
  - context morphisms [Goguen and McKinna '97, Adams '06]
  - CMs extend to syntax translations
- Considerably shorter proofs (here: 3000 loc $\rightsquigarrow$ 700 loc)

Overview

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1 Setup

- de Bruijn Syntax & Parallel Substitutions
- F
- $\lambda 2$
- Equivalence Statement

2 Reductions

- Challenges
- Problem Decomposition
- Context Morphisms – an Example
- Translating Syntax
- Preservation of Typing
- Cancellation Laws

Syntax and Substitutions

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- de Bruijn Syntax: $s, t := n \mid \lambda.s \mid s\ t$
- Parallel Substitutions: $\sigma, \tau : \mathbb{N} \to \mathcal{T} \qquad \sigma = \sigma(0), \sigma(1), \ldots$
  - $t[\sigma]$ applies σ to all free variables of t simultaneously.
  - Cons operation: $s \cdot \sigma := s, \sigma(0), \sigma(1), \ldots$
  - Application $t[\sigma]$ and composition $\sigma \circ \tau$ are mutually recursive:

$$x[\sigma] = \sigma(x) \qquad (\sigma \circ \tau)(x) = \sigma(x)[\tau]$$
$$(s\ t)[\sigma] = s[\sigma]\ t[\sigma]$$
$$(\lambda.s)[\sigma] = \lambda.s[\Uparrow \sigma] \qquad \textbf{where } \Uparrow \sigma := 0 \cdot \sigma \circ +1$$

  - No separate shifting operation, no unnatural lemma statements.
  - Enables the use of context morphisms.
  - Underlying theory: σ-calculus [Abadi et al. '91, Schäfer et al. CPP'15]
    $\Rightarrow$ algebra with computable, unique normal forms.
- Coq library: Autosubst [Schäfer et al. ITP'15].

Variant: F [Harper '13]

Syntax

Separate syntactic sorts for types and terms, $x : \mathbb{N}$:

$$\begin{array}{lrcl} \mathsf{Ty}_F & A, B, C & := & x_{\mathsf{ty}} \mid A \to B \mid \forall.\, A \\ \mathsf{Ter}_F & s, t & := & x_{\mathsf{ter}} \mid s\, t \mid \lambda A.\, s \mid s\, A \mid \Lambda.\, s \end{array}$$

Note: $t[\tau, \sigma]$ denotes the parallel application of both a type substitution $\tau : \mathbb{N} \to \mathsf{Ty}_F$ and a term substitution $\sigma : \mathbb{N} \to \mathsf{Ter}_F$ to the term t.

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Type System

Separate judgements for type formation and typing:

$$N \vdash_{\mathsf{F}}^{\mathsf{ty}} A \qquad\qquad N; \Gamma \vdash_{\mathsf{F}}^{\mathsf{ter}} s : A$$

Variant: $\lambda 2$ [Barendregt '91]

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Syntax

Single syntactic sort, $x : \mathbb{N}$:

$$\mathsf{Ter}_P \qquad a, b, c, d \;:=\; u \mid x \mid a\,b \mid \lambda a.\, b \mid \Pi a.\, b \qquad\qquad u \in \{*, \Box\}$$

Variant: $\lambda 2$ [Barendregt '91]

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Type System

Single judgement:

$$\Gamma \vdash_2 a : b$$

Note: $\Gamma \vdash_2 b : *$ represents type formation.

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Herman Geuvers, '93, doctoral thesis

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Theorem (Reduction of Typing[1])

There are syntax translations $\lfloor\cdot\rfloor$ *and* $\lceil\cdot\rceil$*, such that*

$$\vdash_{\mathsf{F}} s : A \iff \vdash_{2} \lfloor s\rfloor : \lfloor A\rfloor$$
$$\vdash_{2} a : b \iff \vdash_{\mathsf{F}} \lceil a\rceil : \lceil b\rceil$$

[1] A similar result holds for Type Formation.

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$$\vdash_2 a : b \iff \vdash_F \lceil a\rceil : \lceil b\rceil$$

- Requires 2 preservation laws and 2 cancellation laws.
- Contexts can be internalised – so empty context is sufficiently general.

[1] A similar result holds for Type Formation.

Challenges

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2. Different Syntaxes:
   - single-sorted (x) vs. two-sorted (x_{ty} and x_{ter})
   - uniform ($\Pi a.\, b$) vs. non-uniform ($A \to B$ and $\forall.\, A$)
   - different expressivity (prior to typing), e.g. $*, * \vdash_{\mathbf{2}} 1\,0 : 0$.

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$$1; (\forall.\, 0_t\, y) \vdash_{\mathsf{F}}^{\mathsf{ter}} 0_{\mathsf{ter}}\, 0_{\mathsf{ty}} : 0_{\mathsf{ty}}$$

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This is infeasible: $N; \Gamma \vdash_{\mathsf{F}}^{\mathsf{ter}} s : A \;\to\; \lfloor N; \Gamma \rfloor \vdash_2 \lfloor s \rfloor : \lfloor A \rfloor$

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4. Minimal lemma generalisations are often unwieldy.

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4. Minimal lemma generalisations are often unwieldy.

Context Morphisms to the rescue!

Mismatch of Inference Systems

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$$\frac{N; \Gamma, A \vdash_{\mathsf{F}}^{\mathsf{ter}} s : B \qquad N \vdash_{\mathsf{F}}^{\mathsf{ty}} A}{N; \Gamma \vdash_{\mathsf{F}}^{\mathsf{ter}} \lambda A.\, s : A \to B} \qquad \frac{(N+1); \Gamma[+1] \vdash_{\mathsf{F}}^{\mathsf{ter}} s : A}{N; \Gamma \vdash_{\mathsf{F}}^{\mathsf{ter}} \Lambda.\, s : \forall.\, A}$$

$$\frac{\Gamma, a \vdash_2 b : c \qquad \Gamma \vdash_2 a : u \qquad \Gamma, a \vdash_2 c : *}{\Gamma \vdash_2 \lambda a.\, b : \Pi a.\, c}$$

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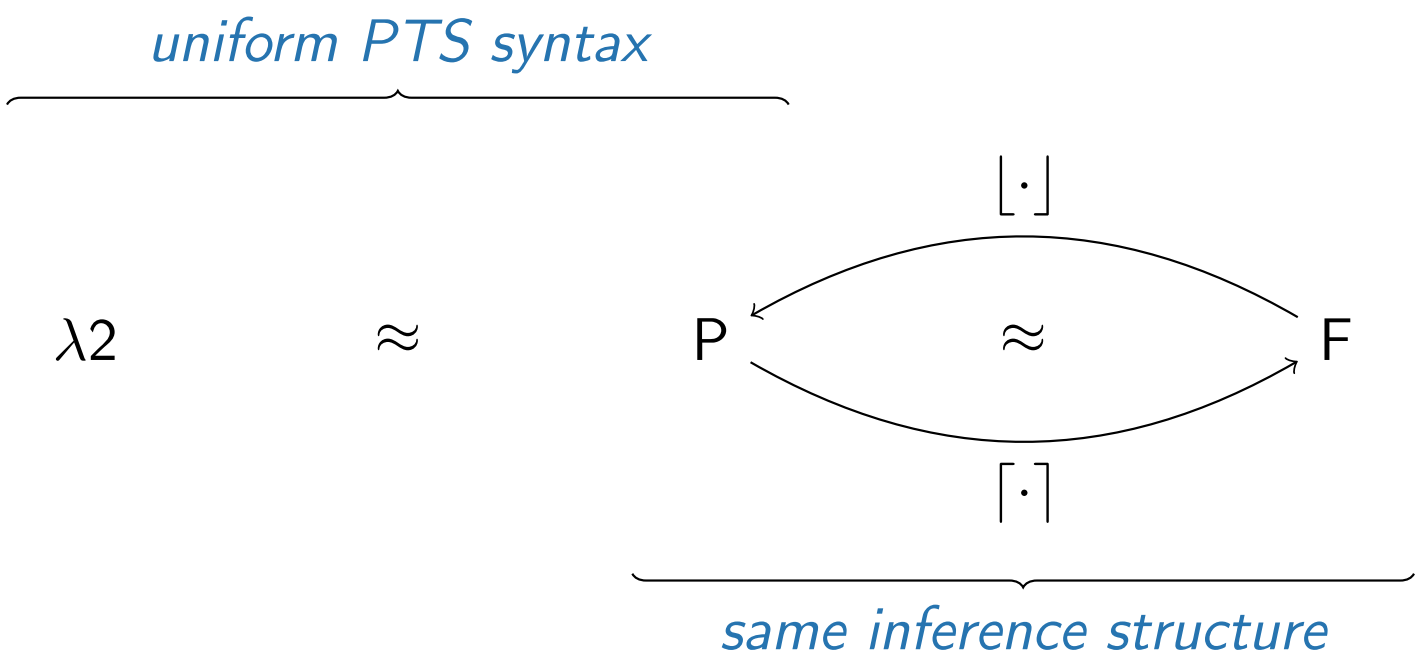
$$\frac{\Gamma, a \vdash_{\mathsf{P}}^{\mathsf{ter}} b : c \qquad \Gamma \vdash_{\mathsf{P}}^{\mathsf{ty}} a}{\Gamma \vdash_{\mathsf{P}}^{\mathsf{ter}} \lambda a.\, b : \Pi a.\, c} \qquad \frac{\Gamma, * \vdash_{\mathsf{P}}^{\mathsf{ter}} a : b}{\Gamma \vdash_{\mathsf{P}}^{\mathsf{ter}} \lambda *.\, a : \Pi *.\, b}$$

$$\frac{\Gamma, a \vdash_2 b : c \qquad \Gamma \vdash_2 a : u \qquad \Gamma, a \vdash_2 c : *}{\Gamma \vdash_2 \lambda a.\, b : \Pi a.\, c}$$

We introduce an auxiliary type system P on the PTS syntax of $\lambda 2$ that matches F.

Problem Decomposition



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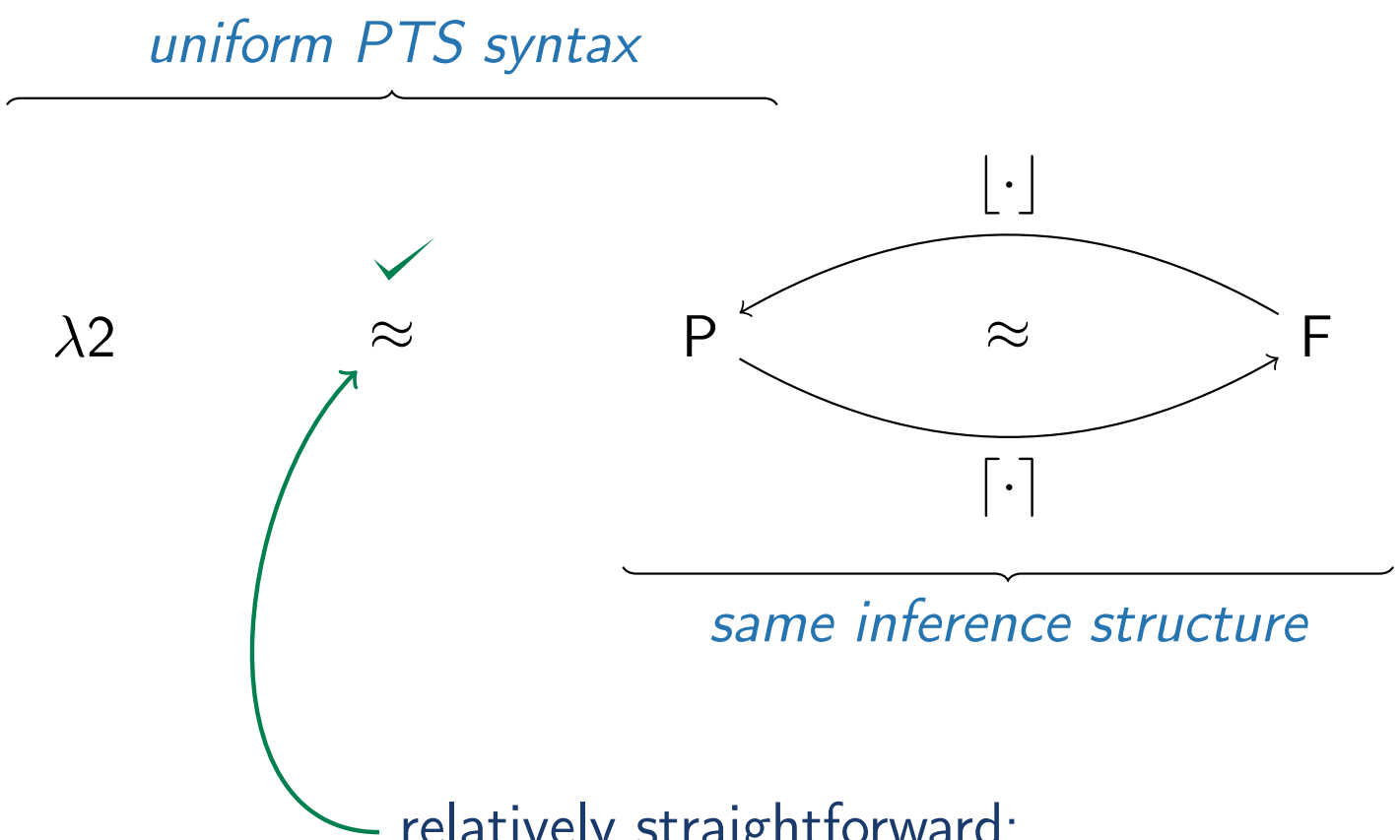

Problem Decomposition

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uniform PTS syntax

$\lambda 2$ $\quad \checkmark \approx$ $\quad$ P $\lfloor \cdot \rfloor$ $\approx$ $\lceil \cdot \rceil$ F

same inference structure

Problem Decomposition

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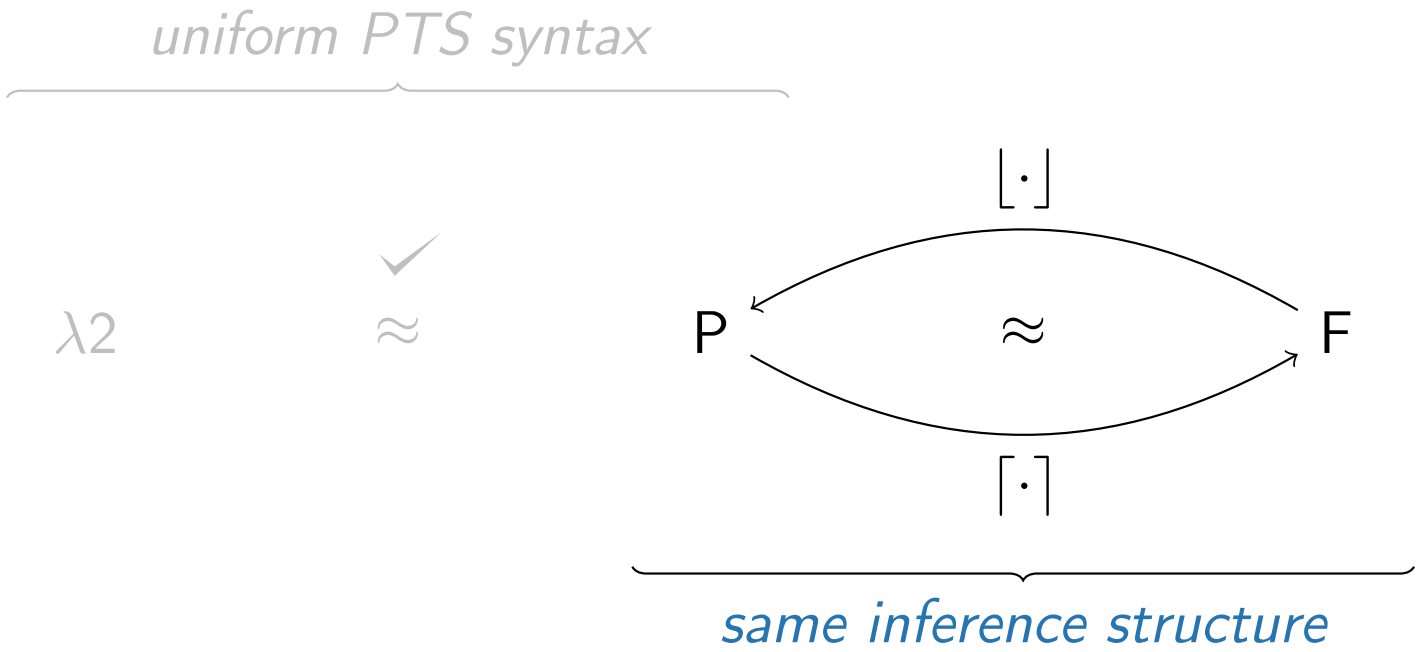

Theorem (PF-Reduction of Typing Problem under Translations)

(1) $\vdash^{\mathsf{ter}}_{\mathsf{F}} s : A \iff \vdash^{\mathsf{ter}}_{\mathsf{P}} \lfloor s \rfloor : \lfloor A \rfloor$

(2) $\vdash^{\mathsf{ter}}_{\mathsf{P}} a : b \iff \vdash^{\mathsf{ter}}_{\mathsf{F}} \lceil a \rceil^{\langle\rangle}_{\mathsf{ter}} : \lceil b \rceil^{\langle\rangle}_{\mathsf{ty}}$

Example: Weakening

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the standard/painful way: minimal generalisation

$$\frac{\Gamma \vdash s : t}{\Gamma, u \vdash s[+1] : t[+1]}$$

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the standard/painful way: minimal generalisation

$$\frac{\Gamma_1, \Gamma_2 \vdash s : t}{\Gamma_1, u, \Gamma_2[\Uparrow^{?}(+1)] \vdash s[\Uparrow^{|\Gamma_2|}(+1)] : t[\Uparrow^{|\Gamma_2|}(+1)]}$$

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$$\frac{\Gamma_1, \Gamma_2 \vdash s : t}{\Gamma_1, u, \Gamma_2[\Uparrow^{?}(+1)] \vdash s[\Uparrow^{|\Gamma_2|}(+1)] : t[\Uparrow^{|\Gamma_2|}(+1)]}$$

- Issue 1: arithmetic, e.g. $x \overset{?}{<} |\Gamma_2|$
- Issue 2: the operation $\Gamma_2[\Uparrow^{?}(+1)]$

let $\Gamma = u_{|\Gamma|-1}, u_{|\Gamma|-2}, \ldots, u_0$

then $\Gamma[\Uparrow^{?}(+1)] = u_{|\Gamma|-1}[\Uparrow^{0}(+1)], u_{|\Gamma|-2}[\Uparrow^{1}(+1)], \ldots, u_0[\Uparrow^{|\Gamma|-1}(+1)]$

Example: Weakening

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the elegant way: maximal generalisation

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Example: Weakening

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the elegant way: maximal generalisation

$$\frac{\Gamma \vdash s : t}{\Delta \vdash s[\sigma] : t[\sigma]}$$

Example: Weakening

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the elegant way: maximal generalisation

$$\frac{\Gamma \vdash s : t \qquad \forall (x : u) \in \Gamma.\ \ \Delta \vdash x[\sigma] : u[\sigma]}{\Delta \vdash s[\sigma] : t[\sigma]}$$

Example: Weakening

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$$\frac{\Gamma \vdash s : t \qquad \forall (x : u) \in \Gamma.\ \ \Delta \vdash x[\sigma] : u[\sigma]}{\Delta \vdash s[\sigma] : t[\sigma]}$$

- Extra premise *quantifies over initial context* Γ.
  $\Rightarrow$ holds vacuously for $\Gamma = \langle\rangle$.
- Fully specifies the behaviour of σ at *relevant variables*.
- Lemma lifts this from variables to *terms*.

Example: Weakening

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Definition (Context Morphism)

$$\vdash \sigma : \Gamma \to \Delta \quad := \quad \forall (x : u) \in \Gamma. \quad \Delta \vdash x[\sigma] : u[\sigma]$$

$$\frac{\Gamma \vdash s : t \qquad \vdash \sigma : \Gamma \to \Delta}{\Delta \vdash s[\sigma] : t[\sigma]}$$

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Concept

Given contexts Γ and Δ and a judgement $\vdash$, we say that a substitution σ is a *context morphism* from Γ to Δ if it maps *variable judgements* under Γ to judgements under Δ, written $\vdash \sigma : \Gamma \to \Delta$.

Example: Weakening

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Key Properties of CMs

$$\frac{}{\vdash +1 : \Gamma \to \Gamma, u} \qquad \frac{\vdash \sigma : \Gamma \to \Delta}{\vdash \Uparrow \sigma : \Gamma, u \to \Delta, u[\sigma]}$$

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$$\frac{}{\vdash +1 : \Gamma \to \Gamma, u} \qquad \frac{\vdash \sigma : \Gamma \to \Delta}{\vdash \Uparrow \sigma : \Gamma, u \to \Delta, u[\sigma]}$$

From F to PTS Syntax

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$\lfloor \cdot \rfloor : \mathrm{Ty}_F \to \mathrm{Ter}_P$

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From F to PTS Syntax

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$\lfloor \cdot \rfloor : \mathrm{Ty}_F \to \mathrm{Ter}_P$

$$\begin{aligned} \lfloor x_{\mathrm{ty}} \rfloor &:= x \\ \lfloor A \to B \rfloor &:= \Pi \lfloor A \rfloor .\ \lfloor B \rfloor [+1] \\ \lfloor \forall .\, A \rfloor &:= \Pi * .\ \lfloor A \rfloor \end{aligned}$$

$\lfloor \cdot \rfloor : \mathrm{Ter}_F \to \mathrm{Ter}_P$

Note: renamings are essential to avoid namespace collision for bound vars.

From F to PTS Syntax

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\begin{aligned}
\lfloor x_{\mathrm{ter}} \rfloor &:= x \\
\lfloor s\ t \rfloor &:= \lfloor s \rfloor\ \lfloor t \rfloor \\
\lfloor \lambda A.\, s \rfloor &:= \lambda \lfloor A \rfloor .\ \lfloor s[+1, \mathrm{id}] \rfloor \\
\lfloor s\ A \rfloor &:= \lfloor s \rfloor\ \lfloor A \rfloor \\
\lfloor \Lambda .\, s \rfloor &:= \lambda * .\ \lfloor s[\mathrm{id}, +1] \rfloor
\end{aligned}
$$

Note: renamings are essential to avoid namespace collision for bound vars.

From PTS to F Syntax

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$\lceil \cdot \rceil^{\Gamma}_{\text{ty}} : \text{Ter}_P \to \text{option Ty}_F$

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From PTS to F Syntax

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$\lceil\cdot\rceil^{\Gamma}_{\text{ty}} : \text{Ter}_P \to \text{option Ty}_F$

$$\lceil x\rceil^{\Gamma}_{\text{ty}} := x_{\text{ty}} \qquad \text{if } x : * \in \Gamma$$
$$\lceil \Pi *.\, a\rceil^{\Gamma}_{\text{ty}} := \forall.\, \lceil a\rceil^{\Gamma,*}_{\text{ty}}$$
$$\lceil \Pi a.\, b\rceil^{\Gamma}_{\text{ty}} := \lceil a\rceil^{\Gamma}_{\text{ty}} \to \lceil b\rceil^{\Gamma,a}_{\text{ty}}\,[\text{-}1]$$

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$\lceil\cdot\rceil^{\Gamma}_{\text{ter}} : \text{Ter}_P \to \text{option Ter}_F$

$$\lceil x\rceil^{\Gamma}_{\text{ter}} := x_{\text{ter}} \qquad \text{if } x : a \in \Gamma, a \neq *$$
$$\lceil a\, b\rceil^{\Gamma}_{\text{ter}} := \lceil a\rceil^{\Gamma}_{\text{ter}}\ \lceil b\rceil^{\Gamma}_{\text{ty}}$$
$$\lceil a\, b\rceil^{\Gamma}_{\text{ter}} := \lceil a\rceil^{\Gamma}_{\text{ter}}\ \lceil b\rceil^{\Gamma}_{\text{ter}}$$
$$\lceil \lambda *.\, a\rceil^{\Gamma}_{\text{ter}} := \Lambda.\, \lceil a\rceil^{\Gamma,*}_{\text{ter}}[\text{id}, \text{-1}]$$
$$\lceil \lambda a.\, b\rceil^{\Gamma}_{\text{ter}} := \lambda\, \lceil a\rceil^{\Gamma}_{\text{ty}}.\, \lceil b\rceil^{\Gamma,a}_{\text{ter}}[\text{-1}, \text{id}]$$

Generalised Context Morphisms

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Concept

Given contexts Γ and Δ and two judgements $\vdash_1$ and $\vdash_2$, we say that a (multi-sorted) substitution $\overline{\sigma}$ is a *generalised context morphism* from Γ to Δ if it maps *variable judgements* under $(\,\Gamma \vdash_1)$ to judgements under $(\,\Delta \vdash_2)$, written $\overline{\sigma} : (\,\Gamma \vdash_1) \to (\,\Delta \vdash_2)$.

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Definition (CM for PF Term-Translation)

$$\xi, \zeta : (\,\Gamma \vdash_{\mathsf{P}}^{\mathsf{ter}}) \to (\,N; \Delta \vdash_{\mathsf{F}}^{\mathsf{ter}}) :=$$
$$\forall x a.\quad \Gamma \vdash_{\mathsf{P}}^{\mathsf{ter}} x : a \;\to\; N; \Delta \vdash_{\mathsf{F}}^{\mathsf{ter}} \zeta(x) : \lceil a \rceil_{\mathsf{ty}}^{\Gamma}[\xi]$$

Generalised Context Morphisms

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Concept

Given contexts Γ and Δ and two judgements $\vdash_1$ and $\vdash_2$, we say that a (multi-sorted) substitution $\overline{\sigma}$ is a *generalised context morphism* from Γ to Δ if it maps *variable judgements* under $(\,\Gamma \vdash_1\,)$ to judgements under $(\,\Delta \vdash_2\,)$, written $\overline{\sigma} : (\,\Gamma \vdash_1\,) \to (\,\Delta \vdash_2\,)$.

Definition (CM for PF Term-Translation)

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Key Properties have to be generalised accordingly, e.g.

$$\frac{\xi, \zeta : (\,\Gamma \vdash_{\mathsf{P}}^{\mathsf{ter}}\,) \to (\,N; \Delta \vdash_{\mathsf{F}}^{\mathsf{ter}}\,)}{\Uparrow\xi, \zeta(0) \cdot \zeta : (\,\Gamma, * \vdash_{\mathsf{P}}^{\mathsf{ter}}\,) \to (\,N+1; \Delta[+1] \vdash_{\mathsf{F}}^{\mathsf{ter}}\,)}$$

Preservation of Typing

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Lemma (Preservation of Typing under Translations)

$$\frac{N; \Delta \vdash^{\mathsf{ter}}_{\mathsf{F}} s : A \qquad \xi : (N \vdash^{\mathsf{ty}}_{\mathsf{F}}) \to (\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}}) \qquad \xi, \zeta : (N; \Delta \vdash^{\mathsf{ter}}_{\mathsf{F}}) \to (\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}})}{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} \lfloor s[\xi, \zeta] \rfloor : \lfloor A \rfloor [\xi]}$$

$$\frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} a : b \qquad \xi : (\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}}) \to (N \vdash^{\mathsf{ty}}_{\mathsf{F}}) \qquad \xi, \zeta : (\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}}) \to (N; \Delta \vdash^{\mathsf{ter}}_{\mathsf{F}})}{N; \Delta \vdash^{\mathsf{ter}}_{\mathsf{F}} \lceil a \rceil^{\Gamma}_{\mathsf{ter}} [\xi, \zeta] : \lceil b \rceil^{\Gamma}_{\mathsf{ty}} [\xi]}$$

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- Proof by structural induction on the respective first premise.

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- Proof by structural induction on the respective first premise.
- Requires a total of 8 morph. rules.

Preservation of Typing

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Lemma (Preservation of Typing under Translations)

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$$\frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} a : b \qquad \xi : (\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}}) \to (N \vdash^{\mathsf{ty}}_{\mathsf{F}}) \qquad \xi, \zeta : (\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}}) \to (N; \Delta \vdash^{\mathsf{ter}}_{\mathsf{F}})}{N; \Delta \vdash^{\mathsf{ter}}_{\mathsf{F}} \lceil a \rceil^{\Gamma}_{\mathsf{ter}} [\xi, \zeta] : \lceil b \rceil^{\Gamma}_{\mathsf{ty}} [\xi]}$$

- Proof by structural induction on the respective first premise.
- Requires a total of 8 morph. rules.
- These settle the forward implications of our main theorem.

Cancellation Laws

$$\frac{\lceil a[\xi] \rceil^{\Gamma}_{\mathsf{ter}} = s}{\lfloor s \rfloor = a[\xi]} \qquad \frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} \lfloor s[\xi, \zeta] \rfloor : c \qquad \Gamma \vdash_{\mathsf{P}} \xi \parallel \zeta}{\lceil \lfloor s[\xi, \zeta] \rfloor \rceil^{\Gamma}_{\mathsf{ter}} = s[\xi, \zeta]}$$

- where $\Gamma \vdash_{\mathsf{P}} \xi \parallel \zeta$ expresses that, according to Γ, ξ only yields type variables and ζ only yields term variables.

Cancellation Laws

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$$\frac{\lceil a[\xi] \rceil^{\Gamma}_{\mathsf{ter}} = s}{\lfloor s \rfloor = a[\xi]} \qquad \frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} \lfloor s[\xi, \zeta] \rfloor : c \qquad \Gamma \vdash_{\mathsf{P}} \xi \parallel \zeta}{\lceil \lfloor s[\xi, \zeta] \rfloor \rceil^{\Gamma}_{\mathsf{ter}} = s[\xi, \zeta]}$$

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- Proof by induction on a and s respectively.

Cancellation Laws

$$\frac{\lceil a[\xi] \rceil^{\Gamma}_{\mathsf{ter}} = s}{\lfloor s \rfloor = a[\xi]} \qquad \frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} \lfloor s[\xi, \zeta] \rfloor : c \qquad \Gamma \vdash_{\mathsf{P}} \xi \parallel \zeta}{\lceil \lfloor s[\xi, \zeta] \rfloor \rceil^{\Gamma}_{\mathsf{ter}} = s[\xi, \zeta]}$$

- where $\Gamma \vdash_{\mathsf{P}} \xi \parallel \zeta$ expresses that, according to Γ, ξ only yields type variables and ζ only yields term variables.
- Proof by induction on a and s respectively.
- *Important:* neither require well-typing for the initial term.

Summary

- Full Reduction of the type formation and typing problems in both directions, formalised in Coq.
- Lessons learned:
  - The standard approach of minimally generalising statements tends to introduce a lot of unnecessary complexity.
  - *De Bruijn syntax*, paired with *parallel substitutions* and *context morphisms* enables clean formalisations of syntax with binders.
  - *Context morphisms* scale to translation scenarios.

Thank you for your attention.

`http://www.ps.uni-saarland.de/extras/hor16`

System F – full definition

$$A, B, C \;:=\; x_{\text{ty}} \mid A \to B \mid \forall.\, A \qquad s, t \;:=\; x_{\text{ter}} \mid s\, t \mid \lambda A.\, s \mid s\, A \mid \Lambda.\, s \qquad x \in \mathbb{N}$$

$$\frac{x < N}{N \vdash_{\mathbf{F}}^{\mathbf{ty}} x_{\text{ty}}} \qquad \frac{N \vdash_{\mathbf{F}}^{\mathbf{ty}} A \qquad N \vdash_{\mathbf{F}}^{\mathbf{ty}} B}{N \vdash_{\mathbf{F}}^{\mathbf{ty}} A \to B} \qquad \frac{(N+1) \vdash_{\mathbf{F}}^{\mathbf{ty}} A}{N \vdash_{\mathbf{F}}^{\mathbf{ty}} \forall.\, A}$$

$$\frac{A_x = A \qquad N \vdash_{\mathbf{F}}^{\mathbf{ty}} A}{N; A_n, \ldots, A_0 \vdash_{\mathbf{F}}^{\mathbf{ter}} x_{\text{ter}} : A}$$

$$\frac{N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} s : A \to B \qquad N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} t : A}{N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} s\, t : B} \qquad \frac{N; \Gamma, A \vdash_{\mathbf{F}}^{\mathbf{ter}} s : B \qquad N \vdash_{\mathbf{F}}^{\mathbf{ty}} A}{N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} \lambda A.\, s : A \to B}$$

$$\frac{N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} s : \forall.\, A \qquad N \vdash_{\mathbf{F}}^{\mathbf{ty}} B}{N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} s\, B : A[B \cdot \mathrm{id}]} \qquad \frac{(N+1); \Gamma[+1] \vdash_{\mathbf{F}}^{\mathbf{ter}} s : A}{N; \Gamma \vdash_{\mathbf{F}}^{\mathbf{ter}} \Lambda.\, s : \forall.\, A}$$

$\lambda 2$ – full definition

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$$a, b, c, d \;:=\; u \mid x \mid a\,b \mid \lambda a.\, b \mid \Pi a.\, b \qquad u \in \{*, \Box\} \qquad x \in \mathbb{N}$$

$$\frac{}{0 : s[+1] \;\in\; \Gamma, s} \qquad \frac{x : s \;\in\; \Gamma}{(x+1) : s[+1] \;\in\; \Gamma, t}$$

$$\frac{}{\Gamma \vdash_2 * : \Box} \qquad \frac{x : a \in \Gamma \qquad \Gamma \vdash_2 a : u}{\Gamma \vdash_2 x : a} \qquad \frac{\Gamma \vdash_2 a : u \qquad \Gamma, a \vdash_2 b : *}{\Gamma \vdash_2 \Pi a.\, b : *}$$

$$\frac{\Gamma \vdash_2 a : \Pi c.\, d \qquad \Gamma \vdash_2 b : c}{\Gamma \vdash_2 a\,b : d[b \cdot \mathrm{id}]} \qquad \frac{\Gamma, a \vdash_2 b : c \qquad \Gamma \vdash_2 a : u \qquad \Gamma, a \vdash_2 c : *}{\Gamma \vdash_2 \lambda a.\, b : \Pi a.\, c}$$

Auxiliary type system P – full definition

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$$\frac{x : * \in \Gamma}{\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} x} \qquad \frac{\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} a \qquad \Gamma, a \vdash^{\mathsf{ty}}_{\mathsf{P}} b}{\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} \Pi a.\, b} \qquad \frac{\Gamma, * \vdash^{\mathsf{ty}}_{\mathsf{P}} a}{\Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} \Pi *.\, a}$$

$$\frac{x : a \in \Gamma \qquad \Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} a}{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} x : a}$$

$$\frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} a : \Pi c.\, d \qquad \Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} b : c}{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} a\, b : d[b \cdot \mathrm{id}]} \qquad \frac{\Gamma, a \vdash^{\mathsf{ter}}_{\mathsf{P}} b : c \qquad \Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} a}{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} \lambda a.\, b : \Pi a.\, c}$$

$$\frac{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} a : \Pi *.\, b \qquad \Gamma \vdash^{\mathsf{ty}}_{\mathsf{P}} c}{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} a\, c : b[c \cdot \mathrm{id}]} \qquad \frac{\Gamma, * \vdash^{\mathsf{ter}}_{\mathsf{P}} a : b}{\Gamma \vdash^{\mathsf{ter}}_{\mathsf{P}} \lambda *.\, a : \Pi *.\, b}$$