Axiomatic Set Theory in Type Theory

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How should we teach axiomatic set theory to students familiar with type theory and proof assistants? Following one of the many textbooks introducing axiomatic set theory does not make sense. In contrast to students of mathematics, our students are familiar with automatic proof checking and an expressive higher-order language for studying axiomatizations. There is no need to talk about the foundations of mathematics. We can just write down the axioms and start proving interesting consequences.

We assume a type theory with excluded middle and an impredicative universe of propositions. We are using the proof assistant Coq. We profit much from Coq's support for inductive definitions und inductive proofs. Given that our inductive definitions concern only predicates, they could be replaced with impredicative definitions.

Let us state the axioms for sets. We assume a type $\mathbf{S}$ and call the members of $\mathbf{S}$ *sets*. A *class* is a predicate $\mathbf{S} \to \mathrm{Prop}$. The letters x, y, and z will range overs sets, and p and q will range over classes. *Inclusion* $p \subseteq q$ and *equivalence* $p \equiv q$ of classes are defined as one would expect. A class is *unique* if it contains at most one set.

We assume a predicate $\in\, : \mathbf{S} \to \mathbf{S} \to \mathrm{Prop}$ for *set membership*. Notationally, we identify a set x with the class $\lambda z.z \in x$. This way notations like $x \subseteq y$, $x \subseteq p$, and $p \equiv x$ are available. A class p is *realizable* if there exists a set x such that $p \equiv x$. It is straightforward to see that the class $\lambda x.x \notin x$ is unrealizable. We assume extensionality of sets:

$$x = y \;\leftrightarrow\; x \equiv y$$

We assume constants $\emptyset$, $\{x, y\}$, $\bigcup x$, $\mathscr{P}x$, and $R@x$ to account for the empty set, unordered pairs, unions, power sets, and replacements. The meaning of the constants is given by the following universally quantified axioms:

$$\begin{aligned}
z \in \emptyset &\;\leftrightarrow\; \bot \\
z \in \{x, y\} &\;\leftrightarrow\; z = x \lor z = y \\
z \in \textstyle\bigcup x &\;\leftrightarrow\; \exists y \in x.\ z \in y \\
z \in \mathscr{P}x &\;\leftrightarrow\; z \subseteq x \\
z \in R@x &\;\leftrightarrow\; \exists y \in x.\ Ryz \land Ry \text{ unique}
\end{aligned}$$

The axioms are known from the set theory ZF. The axiom for replacement $R@x$ is higher-order since it quantifies over a relation $R : \mathbf{S} \to \mathbf{S} \to \mathrm{Prop}$. In first-order logic, replacement can only be expressed with a scheme describing infinitely many axioms. Our formulation of the replacement axiom is equivalent to a more conventional formulation requiring R to be functional.

For the results we claim in this abstract we do not need the axioms for infinity, choice, and regularity. For the axioms given one can construct a model in type theory based on Ackermann's encoding [1] of hereditarily finite sets. Thus the axioms given for sets do not affect consistency. For the general case with infinity and choice, consistency has been studied by Aczel [3] and Werner [4].

Singletons and binary unions of sets can be expressed as one would expect: $\{x\} := \{x, x\}$ and $x \cup y := \bigcup\{x, y\}$. Operators for separation $x \cap p$ and description $\ulcorner p \urcorner$ (obtaining the element of a singleton class) can be expressed with replacement.

We define the class $\mathscr{W}$ of *well-founded sets* inductively with a single rule:

$$\frac{x \subseteq \mathscr{W}}{x \in \mathscr{W}}$$

This kind of definition is familiar in type theory but unknown in first-order presentations of set theory. Note that $\mathscr{W}$ defines well-founded sets as sets allowing for epsilon induction (a notion known in first-order set theory).

The regularity axiom says that every set is well-founded. While it is easy to express the regularity axiom as a first-order formula, expressing well-foundedness of a single set is difficult; it seems that the infinity axiom is needed so that transitive closure of sets can be expressed. The limitations of first-order logic may be the reason that ZF comes with the regularity axiom disallowing non-well-founded sets. There is agreement that the regularity axiom is not needed mathematically. There is also Aczel's non-well-founded set theory [2] that has as an axiom postulating the existence of non-well-founded sets.

Next we define that class $\mathscr{Z}$ of *cumulative sets* inductively by means of two rules:

$$\frac{x \subseteq \mathscr{Z}}{\bigcup x \in \mathscr{Z}} \qquad\qquad \frac{x \in \mathscr{Z}}{\mathscr{P}x \in \mathscr{Z}}$$

The class $\mathscr{Z}$ is known as *cumulative hierarchy* or as *Zermelo hierarchy* or as *von Neumann hierarchy* in the literature. In the literature, $\mathscr{Z}$ is defined by transfinite recursion on ordinals. We have not seen a definition of $\mathscr{Z}$ not making use of the ordinals. Here are the main results we have shown for $\mathscr{W}$ and $\mathscr{Z}$:

1. $\mathscr{W}$ and $\mathscr{Z}$ are unrealizable.
2. $\mathscr{W} \equiv \bigcup \mathscr{Z}$.
3. $\mathscr{Z}$ is well-ordered by set inclusion.
4. For every well-ordered set x there exists a unique cumulative set z such that x and the set $\{\, y \in \mathscr{Z} \mid y \subset z \,\}$ are order isomorphic ($\{\, y \in \mathscr{Z} \mid y \subset z \,\}$ is ordered by set inclusion).

Result (4) says that the cumulative sets can serve as unique set representations of the isomorphism classes of well-ordered sets. In the literature, von Neumann ordinals are introduced for this purpose.

The theory of well-orderings and transfinite recursion should be developed generally in pure type theory. It can then be applied to the axiomatized system of sets. For the results mentioned in this abstract neither transfinite recursion nor ordinals are needed.

The Coq development accompanying this abstract has been carried out by Dominik Kirst. Together with a full paper it can be found at `https://www.ps.uni-saarland.de/extras/types15`.

References

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