Axiomatic Set Theory in Type Theory

Gert Smolka

Saarland University
May 15, 2015

1 Introduction

Axiomatic set theory [14, 5, 7] has been developed based on first-order logic. We study the possibilities opening up when axiomatic set theory is developed in type theory with excluded middle. We arrive at the conclusion that type theory is much better suited for the development of set theory than first-order logic:

- Many notions of set theory, for instance, well-orderings and transfinite recursion, can be developed more generally and more naturally in type theory. They can then be applied to sets (i.e., the elements of an axiomatized type of sets).
- Zermelo-Fraenkel set theory cannot be formalized in pure first-order logic. An extension providing for axiom schemes is needed to express and apply the replacement axiom. In type theory, finite axiomatizations of set theories are obvious since one can quantify over relations. Another difficulty is transfinite recursion, where one often wants functions at the class level (e.g., ordinal arithmetic).
- Type theory provides an expressive logic for the class level of set theory. The set-theoretic classes comprised of well-founded sets, ordinals, and cumulative sets have elegant inductive definitions in type theory. General results about inductive constructions in type theory can be applied to obtain more specific set-theoretic results.
- The development of an axiomatic set theory profits much from the use of a proof assistant. Powerful proof assistants are available for type theory. We are using the proof assistant Coq.

We are working in a constructive type theory with an impredicative universe of propositions (calculus of constructions [8]). We make free use of excluded middle for propositions. We frequently use inductive predicates and profit from Coq's support for inductive proofs. Our inductive definitions could be replaced with plain impredicative definitions. We do not use inductive types and do not assume extensionality for predicates or functions.

We are confident that the logical system we are considering (calculus of constructions plus excluded middle plus an axiomatized type of sets) is consistent. Our confidence rests on work of Aczel [3], Werner [15], and others who interpret set theory in type theory and vice versa.

Developing axiomatic set theory in type theory is not a new idea. Barras [4] presents a substantial development of axiomatic intuitionistic set theory in type theory and Coq.

We are interested in a high-level development of classical set theory in type theory. We do not follow a conventional text on set theory (e.g. [14, 5, 7]) but are driven by questions of the following kind:

- Which parts of set theory can be developed more generally in pure type theory?
- How should we organize set theory given the expressive means of type theory?
- How should we teach set theory to students familiar with type theory?

We think that doing classical set theory in type theory is very helpful in understanding the foundations of mathematics. We found much mathematical beauty in this project.

The main contribution of the paper is a presentation of the theory of ordinals and the cumulative hierarchy making full use of the expressive power of type theory. A distinctive feature of our presentation is the systematic use of inductive predicate definitions. We show that the theory of well-orderings can be developed more generally and more naturally in pure type theory before it is applied to an axiomatized type of sets.

The Coq development accompanying this paper has been carried out by Dominik Kirst and can be found at `https://www.ps.uni-saarland.de/extras/types15`. Dominik contributed Theorem 46 presented in the paper.

2 Classes and Relations

Let X and Y be types. A **class on X** is a unary predicate $X \to \text{Prop}$, a **relation on X** is a binary predicate $X \to X \to \text{Prop}$, and a **relation from X to Y** is a binary predicate $X \to Y \to \text{Prop}$. An **identity on X** is a relation R on X such that $x = y$ whenever Rxy. Classes and relations play an important role in the type-theoretic study of sets. We are not interested in the set-theoretic representation of relations.

A class p is **inhabited** if px for some x. A class is **empty** if it is not inhabited. A class p is **unique** if $x = x'$ whenever px and px'. We write $\mathscr{C}X := X \to \text{Prop}$ for the type of classes over X and use notations familiar from sets:

$$x \in p := px \qquad \{\, x \mid s \,\} := \lambda x.s$$
$$p \subseteq q := \forall x \in p.\ x \in q \qquad p \equiv q := p \subseteq q \wedge q \subseteq p$$
$$p \subset q := p \subseteq q \wedge \exists x \in q.\ x \notin p$$

Let R and R' be relations from X to Y. We define:

$$\begin{aligned} R^{-1} &:= \lambda xy.\ Ryx && \textbf{inverse of } R \\ \mathscr{D}R &:= \lambda x.\ \exists y.Rxy && \textbf{domain of } R \\ R \subseteq R' &:= \forall x.\ Rx \subseteq R'x && \textbf{inclusion} \\ R \equiv R' &:= R \subseteq R' \wedge R' \subseteq R && \textbf{equivalence} \end{aligned}$$

Note that $\mathscr{D}R \subseteq \mathscr{D}R'$ if $R \subseteq R'$. We call R **functional** if Rx is unique for every x. We call R a **bijection** if R and R^{-1} are functional.

3 Sets

We assume a type $\mathbf{S}$ and call the members of $\mathbf{S}$ **sets**. We also assume a predicate $\in : \mathbf{S} \to \mathbf{S} \to \mathrm{Prop}$ for **set membership**. The letters x, y, and z will range overs sets. Notationally, we identify a set x with the class $\lambda z.z \in x$. This way notations like $x \subseteq y$, $x \subseteq p$, and $p \equiv x$ are available. We assume extensionality of sets:

- $x = y$ whenever $x \equiv y$.

We assume constants $\emptyset$, $\{x, y\}$, $\bigcup x$, $R@x$, and $\mathscr{P}x$ to account for the description of the empty set, unordered pairs, unions, replacements, and power sets (R ranges over relations on $\mathbf{S}$). The meaning of the constants is given by the following universally quantified axioms:

- $z \notin \emptyset$.
- $z \in \{x, y\}$ iff $z = x$ or $z = y$.
- $z \in \bigcup x$ iff $z \in y \in x$ for some y.
- $z \in R@x$ iff Ryz, $y \in x$, and Ry is unique for some y.
- $z \in \mathscr{P}x$ iff $z \subseteq x$.

The axioms are well-known from the set theory ZF. The axiom for replacement $R@x$ deviates from the standard formulation in that it does not require R to be functional; instead only uniquely determined images of elements of x are collected. Singletons and binary unions of sets are expressed as one would expect:

$$\begin{aligned} \{x\} &:= \{x, x\} \\ x \cup y &:= \bigcup \{x, y\} \end{aligned}$$

The most interesting set constructor is replacement, which differs from the other constructors in that it is higher-order. Replacement can express separation and description. Let $f : \mathbf{S} \to \mathbf{S}$ and $p : \mathscr{C}\,\mathbf{S}$. We define:

$$\begin{aligned} f@x &:= (\lambda yz.\, z{=}fy)@x && \textbf{functional replacement} \\ x \cap p &:= (\lambda yz.\, z{=}y \wedge pz)@x && \textbf{separation} \\ \ulcorner p \urcorner &:= \bigcup((\lambda y.\, p)@\{\emptyset\}) && \textbf{description} \end{aligned}$$

We will take the freedom to write $\{\, z \in x \mid pz \,\}$ for $x \cap p$.

Fact 1 Let $f : \mathbf{S} \to \mathbf{S}$ and $p : \mathscr{C}\,\mathbf{S}$.

- $z \in f@x$ iff $z = fy$ for some $y \in x$.
- $z \in x \cap p$ iff $z \in x$ and pz.
- $p\ulcorner p \urcorner$ if p is inhabited and unique.

For the results of this paper, we do not need axioms for regularity, infinity, and choice. Since we do not impose regularity, our results apply to both well-founded and non-well-founded set theory [2].

We can construct in type theory a model for our axiomatized type of sets using Ackermann's encoding [1] of hereditarily finite sets, provided we assume strong excluded middle (type-level decidability of type inhabitation).

For the results of this paper, we can omit the constant and the axiom for the empty set (see the Coq development). The empty set is only used in the above definition of description, which is not used in the rest of the paper. Without the empty set, the empty type yields a trivial model of our axioms for sets.

We define the class of **transitive sets** as $\mathscr{T} := \lambda x.\, \forall y \in x.\, y \subseteq x$

Fact 2 $x \in \mathscr{T}$ iff $\bigcup x \subseteq x$ iff $x \subseteq \mathscr{P}x$.

Let p be a class of sets. We call p **realizable** if $p \equiv x$ for some set x. We call p **transitive** if $x \subseteq p$ if $x \in p$. We call p **cumulative** if it satisfies the following properties:

1. If $x \in y \in p$, then $x \subset y$.
2. If $x, y \in p$ and $x \subset y$, then $x \in y$.

Fact 3 Every superclass of an unrealizable class is unrealizable.

Fact 4 Every cumulative class is a subclass of $\mathscr{T}$.

4 Well-Founded Sets

We define the class $\mathscr{W}$ of **well-founded sets** inductively with a single rule:

$$\frac{x \subseteq \mathscr{W}}{x \in \mathscr{W}}$$

This kind of inductive definition is familiar in type theory but unknown in first-order set theory. In Zermelo Fraenkel set theory, the class of well-founded sets is usually written V and is obtained as the union of all stages V_α of the cumulative hierarchy obtained with transfinite recursion from the von Neumann ordinals.

The induction principle for $\mathscr{W}$ is known as epsilon induction in set theory. In fact, we have defined $\mathscr{W}$ as the class of all sets allowing for epsilon induction.

The regularity axiom says that every set is well-founded. Given the definition of $\mathscr{W}$, we can state the regularity axiom of ZF as $\forall x.\, x \in \mathscr{W}$. Note that epsilon induction can be used for well-founded sets independently of whether or not regularity is assumed.

While it is easy to express the regularity axiom as a first-order formula, expressing well-foundedness of a single set is difficult; it seems that the infinity axiom is needed so that transitive closure of sets can be expressed. Difficulties with the first-order approach may have been the reason that ZF adopts the regularity axiom disallowing non-well-founded sets. There is agreement that the regularity axiom is not needed mathematically. There is also Aczel's non-well-founded set theory [2] that has as an axiom postulating the existence of non-well-founded sets.

Fact 5 $\mathscr{W}$ is transitive and unrealizable.

A set x is **regular** if either x is empty or x contains an element disjoint from x (i.e., $\exists y \in x\, \forall z \in x.\, z \notin y$).

Fact 6 A transitive set x is well-founded if and only if every subset of x is regular.

5 Overview: Ordinals and Cumulative Sets

We define two classes $\mathscr{O}$ and $\mathscr{Z}$ of sets inductively:

$$\frac{x \subseteq \mathscr{O}}{\bigcup x \in \mathscr{O}} \qquad \frac{x \in \mathscr{O}}{x \cup \{x\} \in \mathscr{O}} \qquad\qquad \frac{x \subseteq \mathscr{Z}}{\bigcup x \in \mathscr{Z}} \qquad \frac{x \in \mathscr{Z}}{x \cup \mathscr{P}x \in \mathscr{Z}}$$

The members of $\mathscr{O}$ and $\mathscr{Z}$ are called **ordinals** and **cumulative sets**, respectively. The classes are well-known in set theory, with different but equivalent definitions. Ordinals are usually obtained as transitive sets that are strictly well-ordered by membership. Cumulative sets are usually obtained from ordinals by transfinite recursion. Our inductive definition of cumulative sets is independent of ordinals.[1] Note that the definitions of $\mathcal{O}$ and $\mathcal{Z}$ differ only in the successor operation used in the second rule. One can show that the functions $\lambda x.\, x \cup \{x\}$ and $\lambda x.\, x \cup \mathcal{P}x$ agree with the order-theoretic successor functions for $\mathcal{O}$ and $\mathcal{Z}$. The class $\mathcal{Z}$ is known as *cumulative hierarchy* or as *Zermelo hierarchy* or as *von Neumann hierarchy* in the literature.

We will show the following results:

1. $\mathcal{O}$ and $\mathcal{Z}$ are unrealizable and cumulative subclasses of both $\mathcal{W}$ and $\mathcal{T}$.
2. $\mathcal{O}$ and $\mathcal{Z}$ are well-ordered by set inclusion and are order-isomorphic.
3. A set is well-founded if and only if it is an element of some cumulative set.
4. Every ordinal is the set of all strictly smaller ordinals.
5. Every well-ordered set is order-isomorphic with a unique ordinal.

The results are well-known for the usual development of ordinals and cumulative sets in set theory.

An advantage of the inductive definitions of the classes $\mathcal{O}$ and $\mathcal{Z}$ over the usual definitions is that they come with useful induction principles. With the standard definitions, the induction principles have to be established by lemmas.

6 Well-Orderings

Our proofs of the above results are based on purely type-theoretic results for well-orderings. The fact that we need well-ordering of classes indicates that the notion of well-ordering is more general than the notion of set. The type of sets is not needed for this section.

A well-ordering is a linear ordering such that every inhabited class has a least element. The most important result we need about well-orderings is that given two well-orderings, one of them is order-isomorphic to a section of the other.

For the following definitions we assume that X is a type, R is a relation on X, and p is a class on X. We refer to the elements of X as **points**. We say that

- R is **reflexive** if Rxx and Ryy whenever Rxy.
- R is **antisymmetric** if $x = y$ whenever Rxy and Ryx.
- R is **transitive** if Rxz whenever Rxy and Ryz.
- R is **linear** if for all $x, y \in \mathcal{D}R$ either Rxy or Ryx.
- R is a **partial ordering** if R is reflexive, transitive, and antisymmetric.

[1] Our inductive definition of the class of ordinals is similar to an inductive definition given by Forster [6]. Forster's inductive definition is carried out in an informal setting. He speaks of rectypes and restricts the union rule to chain-ordered subsets.

- R is a **linear ordering** if R is a linear partial ordering.

We define the following predicates:

$$\begin{aligned} L^R p &:= \{\, x \in p \mid \forall y \in p.\, Rxy \,\} && \textbf{least points} \\ \Sigma^R &:= \{\, p \mid p \subseteq \mathscr{D}R \wedge \forall xy.\, Rxy \to py \to px \,\} && \textbf{segments} \\ \Sigma^R_x &:= \{\, z \mid Rzx \wedge z \neq x \,\} && \textbf{segment for } x \\ R|p &:= \lambda xy.\, Rxy \wedge px \wedge py && \textbf{restriction} \end{aligned}$$

We say that a partial ordering R is a **well-ordering** if $L^R p$ is inhabited for every inhabited class $p \subseteq \mathscr{D}R$. We say that $R|p$ is a **section of R** if $p \in \Sigma^R$. A section $R|p$ of R is called **proper** if $p \subset \mathscr{D}R$. Analogously, a segment p of R is called **proper** if $p \subset \mathscr{D}R$.

Fact 7 Let R be a linear ordering and $p, q \in \Sigma^R$. Then either $p \subseteq q$ or $q \subseteq p$.

Fact 8 Let R be a well-ordering. Then:

1. R is a linear ordering.
2. $R|p$ is a well-ordering for every p.
3. If $p \in \Sigma^R$, then either $p \equiv \mathscr{D}R$ or $p \equiv \Sigma^R_x$ for some $x \in \mathscr{D}R$.

Well-orderings come with an induction principle known as well-founded induction. Given a relation R on X, we obtain an induction principle for R with the inductively defined class W^R:

$$\frac{\Sigma^R_x \subseteq W^R}{x \in W^R}$$

The class W^R identifies the points of X to which the induction principle for R applies. Since there is only one defining rule for W^R, we have $x \in W^R$ iff $\Sigma^R_x \subseteq W^R$. We say that R is **well-founded** if $\mathscr{D}R \subseteq W^R$. A well-founded relation provides well-founded induction for every point of its domain.

Fact 9 (Well-Founded Induction) Every well-ordering is well-founded.

Fact 10 A linear ordering is a well-ordering if and only if it is well-founded.

Fact 11 (Acyclicity) Let R be a well-founded relation and $Rx_0x_1, \ldots, Rx_nx_{n+1}$. Then $x_0, \ldots, x_{n+1}$ are pairwise distinct.

Fact 12 Membership on well-founded sets is well-founded.
That is, $\lambda xy.\, x \in y \in \mathscr{W}$ is a well-founded relation.

7 Similarities

For our results we need isomorphisms between well-orderings. In their general form isomorphisms between well-orderings are known as similarities.

Let R be a relation on X, S be a relation on Y, and U be a relation from X to Y. We say that U is a **simulation** if $\mathscr{D}U \subseteq \mathscr{D}R$ and

$$Uxy \to Rx'x \to \exists y'.\, Ux'y' \wedge Sy'y$$

for all x, y, x'. We say that U is a **similarity from R to S** if U and U^{-1} are functional simulations. Note that the domain of a simulation is a segment of R. Also note that similarities are bijections and that U is a similarity if and only if U^{-1} is a similarity.

We say that a similarity U from R to S is an **isomorphism from R to S** if $\mathscr{D}U \equiv \mathscr{D}R$ and $\mathscr{D}(U^{-1}) \equiv \mathscr{D}S$. We say that R and S are **isomorphic** if there exists an isomorphism from R to S.

We say that a similarity U from R to S is **maximal** if either $\mathscr{D}U \equiv \mathscr{D}R$ or $\mathscr{D}(U^{-1}) \equiv \mathscr{D}S$. Note that every isomorphism is maximal.

Fact 13 Let R and S be well-orderings and U be a similarity from R to S. Then:

1. $\mathscr{D}U$ is a segment of R and $\mathscr{D}(U^{-1})$ is a segment of S.
2. U is an isomorphism from $R|\mathscr{D}U$ to $S|\mathscr{D}(U^{-1})$.

Fact 14 (Agreement) Let R and S be well-orderings and U and V be similarities from R to S. Then:

1. If Uxy and Vxy', then $y = y'$.
2. Either $U \subseteq V$ or $V \subseteq U$.
3. If V is maximal, then $U \subseteq V$.
4. If U and V are maximal, then $U \equiv V$.

Proof Claim 1 follows by induction on $x \in W^R$. For Claim 2, note that either $\mathscr{D}R \subseteq \mathscr{D}S$ or $\mathscr{D}S \subseteq \mathscr{D}R$ by Facts 7 and 13. Thus either $U \subseteq V$ or $V \subseteq U$ by Claim 1. Claim 3 follows from Claim 2. ■

Fact 15 Let R and S be well-orderings. Then:

1. There exists a maximal similarity from R to S (unique up to equivalence).
2. If U is a similarity from R to S such that $\mathscr{D}U \equiv \mathscr{D}R$, then U is an isomorphism from R to a section of S.
3. Either R is isomorphic to a section of S or S is isomorphic to a section of R.

Fact 16 Let R be a well-ordering and S and S' be sections of R. Then S and S' are isomorphic if and only if $S \equiv S'$.

8 Well-Orderings on Sets

Let R be a relation on **S**. We call R **realizable** if $\mathscr{D}R$ is realizable. We call R **complete** if R is unrealizable and every proper section of R is realizable. We say that R is a **well-ordering of x** if $\mathscr{D}R \equiv x$.

Fact 17 Let R and S be well-orderings on **S**.

1. If R and S are isomorphic, then R is realizable iff S is realizable.
2. If R and S are complete, then a similarity from R to S is an isomorphism from R to S if $\mathscr{D}U \equiv \mathscr{D}R$.
3. If R and S are complete, then R and S are isomorphic.
4. If R is realizable and S is unrealizable, then there exists a unique $x \in \mathscr{D}S$ such that R is isomorphic with $S|\Sigma^S_x$ (a proper section of S).

Proof *Claim 1.* Suppose U is an isomorphism from R to S and R is realizable. Then $\mathscr{D}R \equiv x$ for some set x. Thus $\mathscr{D}S \equiv U@x$. Hence S is realizable.

Claim 2. Let R and S be complete and U be a similarity from R to a S such that $\mathscr{D}U \equiv \mathscr{D}R$. By Fact 15 we know that U is an isomorphism from R to a section of S. Since R and S are complete, we know by Claim 1 that U is an isomorphism from R to S.

Claim 3. By Fact 15 we have a maximal similarity U from R to S. Without loss of generality, we have $\mathscr{D}U \equiv \mathscr{D}R$. Thus U is an isomorphism from R to S by Claim 2.

Claim 4. Follows from Claim 1 and Facts 15 and 16. ∎

Let p be a class on **S**. We define the **inclusion ordering of p** as

$$I_p := \lambda xy.\, x \subseteq y \wedge px \wedge py$$

We will write Σ^p and Σ^p_x for Σ^{I_p} and $\Sigma^{I_p}_x$. We call p **well-ordered** if I_p is a well-ordering. We call p **linear** if for all $x, y \in p$ either $x \subseteq y$ or $y \subseteq x$.

Fact 18 Let p be a class on **S**. Then:

1. I_p is a partial ordering.
2. $\Sigma^p_x = \{\, y \in \mathscr{P}x \mid px \wedge py \wedge y \neq x \,\}$.
3. Σ^p_x is realizable.
4. I_p is complete iff p is unrealizable.

Fact 19 Let p be a well-ordered class on **S**. Then every proper segment and every proper section of I_p is realizable. Moreover, I_p is complete iff p is unrealizable.

Proof Follows with Facts 18 and 8. ∎

Fact 20 Let p be a linear and cumulative subclass of $\mathscr{W}$. Then p is well-ordered.

Proof Since p is linear, I_p is a linear ordering. By Fact 10 it suffices to show that I_p is well-founded. This follows from Fact 12 and the assumption that p is a cumulative subclass of $\mathscr{W}$. ∎

Fact 21 (Segment Property) Let p be a cumulative class on **S**. Then:

1. $\Sigma_x^p \subseteq x$.
2. $\Sigma_x^p \equiv x = \{\, y \in p \mid y \subset x \,\}$ if $x \in p$ and p is transitive.

9 Towers

We now say more about the inductive construction used for the definition of the classes $\mathscr{O}$ and $\mathscr{Z}$. A recent paper [13] studying the construction in generalized form in pure type theory calls it *tower construction*. We borrow two results from this paper. We consider the tower construction for the special case where it yields a class of sets.

Assumption 22 Let f be a function $\mathbf{S} \to \mathbf{S}$.

We call f **increasing** if $x \subseteq fx$ for all x. We call f **cumulative** if $x \in fx$ for all x. Given f, we define a class T over **S** inductively as follows:

$$\frac{x \subseteq T}{\bigcup x \in T} \qquad\qquad \frac{x \in T}{fx \in T}$$

We say that T is the **tower** obtained with the **step function** f. We begin with two results for towers we borrow from the paper studying the general tower construction [13].

Fact 23 (Successor) Let f be increasing and $x, y \in T$. Then $fx \subseteq y$ if $x \subset y$.

Fact 24 (Linearity) T is linear if f is increasing.

The remaining results we need for towers are easy to prove.

Fact 25 (Transitivity) If f preserves transitivity, then $T \subseteq \mathscr{T}$.

Fact 26 (Well-Foundedness) If f preserves well-foundedness, then $T \subseteq \mathscr{W}$.

Fact 27 (Cumulativity) Let f be increasing and cumulative, and let f preserve transitivity and well-foundedness. Then T is a cumulative subclass of $\mathscr{W}$.

Proof By Fact 26 we know $T \subseteq \mathscr{W}$. We show that T is cumulative.

Let $x \in y \in T$. Then $x \subset y$ since y is transitive and well-founded.

Let $x, y \in T$ and $x \subset y$. Then $x \in fx \subseteq y$ by Fact 23 since f is increasing and cumulative. ∎

Fact 28 (Unrealizability) Let f be increasing and cumulative, and let f preserve well-foundedness. Then T is unrealizable.

Proof By contradiction. Suppose $T \equiv x$. Then $\bigcup x \in x$ and $f(\bigcup x) \in x$. Thus $f(\bigcup x) \subseteq \bigcup x$. Since f is increasing we have $\bigcup x \subseteq f(\bigcup x)$. Thus $\bigcup x = f(\bigcup x)$. Hence $\bigcup x \in \bigcup x$ since f is cumulative. Contradiction by Fact 11 since $\bigcup x \in T$ is well-founded by Fact 26. ∎

Fact 28 may be seen as an abstract version of the Burali-Forti paradox.

Theorem 29 (Tower) Let f be increasing and cumulative, and let f preserve transitivity and well-foundedness. Then T is a well-ordered, cumulative, and unrealizable subclass of $\mathscr{W}$.

Proof By Facts 27 and 28 we know that T is a cumulative, and unrealizable subclass of $\mathscr{W}$. That T is well-ordered now follows with Facts 20 and 24. ∎

10 Ordinals

We now prove the results stated in Section 5 for the inductively defined class $\mathscr{O}$ of ordinals. Given that $\mathscr{O}$ is defined as a tower, we can use the results we have obtained for towers. The remaining proofs are straightforward.

Fact 30 $\mathscr{O} \equiv T(\lambda x. x \cup \{x\})$. The step function $\lambda x. x \cup \{x\}$ preserves transitivity and well-foundedness and is increasing and cumulative.

Theorem 31 $\mathscr{O}$ is a well-ordered, cumulative, and unrealizable subclass of $\mathscr{W}$.

Proof Follows with Theorem 29 and Fact 30. ∎

Fact 32 (Successor) Let $x, y \in \mathscr{O}$ and $x \subset y$. Then $x \cup \{x\} \subseteq y$.

Proof Follows with Facts 23 and 30. ∎

Fact 33 (Transitivity) $\mathscr{O}$ is transitive.

Proof Let $x \in \mathscr{O}$. We prove $x \subseteq \mathscr{O}$ by induction of $x \in \mathscr{O}$.

Let $x \in \mathscr{O}$ such that $x \subseteq \mathscr{O}$ (inductive hypothesis). Then $x \cup \{x\} \subseteq \mathscr{O}$.

Let $x \subseteq \mathscr{O}$ such that $y \subseteq \mathscr{O}$ for all $y \in x$ (inductive hypothesis). Then $\bigcup x \subseteq \mathscr{O}$. ∎

Fact 34 (Segment Property) Every ordinal is the set of all smaller ordinals. That is, $x = \{ y \in \mathcal{O} \mid y \subset x \} \equiv \Sigma_x^{\mathcal{O}}$ if $x \in \mathcal{O}$.

Proof Follows with Fact 21 from Theorem 31 and Fact 33. ∎

Theorem 35 (Completeness) Let R be a realizable well-ordering on sets. Then there exists a unique ordinal x such that R and I_x are isomorphic.

Proof Follows with Theorem 31 and Facts 17 and 34. ∎

Fact 36 (Epsilon Images) Let R be a well-ordering of a set A and let U be a similarity from R to $I_{\mathcal{O}}$. Then $U\,x\,(U@\{ y \in A \mid Ryx \wedge y \neq x \})$ for every $x \in \mathcal{D}U$.

Proof Let Uxu and $v := U@\{ y \in A \mid Ryx \wedge y \neq x \}$. We show $u = v$ by extensionality.

Let $z \in u$. Then $z \subset u$ since $u \in \mathcal{O}$ and $\mathcal{O}$ is cumulative. Since U is a similarity, we have some $y \in A$ such that Ryx, $y \neq x$, and Uyz. Thus $z \in v$.

Let $z \in v$. Then we have some $y \in A$ such that Ryx, $y \neq x$, and Uyz. Then $z \in \mathcal{O}$ and $z \subset u$ since U is a similarity. Thus $z \in u$ since $\mathcal{O}$ is cumulative. ∎

We conclude with some historical remarks about ordinals. Cantor conceived ordinals as abstract objects representing isomorphism classes of well-ordered sets. He did not think of ordinals as sets. It took until 1923 that von Neumann [9] devised the now standard representation of ordinals as sets. In his 1923 paper, von Neumann obtains ordinals by means of transfinite recursion from well-orderings with the scheme expressed by Fact 36. In 1928, von Neumann [10] gave a more explicit definition of ordinals as the domains of well-orderings satisfying the segment property expressed by Fact 34. An explicit definition of ordinals as plain sets avoiding the notion of well-ordering was first given in 1937 by Robinson [11] (see Section 12). Enderton's textbook [5] introduces the ordinals based on von Neumann's 1923 definition (Enderton speaks of the epsilon image of a well-ordering).

11 Cumulative Hierarchy

We now consider the class $\mathcal{Z}$ defined in Section 5. Again we exploit the results we have obtained for towers.

Fact 37 $\mathcal{Z} \equiv T(\lambda x.\, x \cup \mathcal{P}x)$. The step function $\lambda x.\, x \cup \mathcal{P}x$ preserves transitivity and well-foundedness and is increasing and cumulative.

Theorem 38 $\mathcal{Z}$ is a well-ordered, cumulative, and unrealizable subclass of $\mathcal{W}$.

Proof Follows with Theorem 29 and Fact 37. ∎

Fact 39 Let $x \in \mathscr{Z}$. Then x is transitive and $\mathscr{P}x = x \cup \mathscr{P}x$.

Proof Follows with Facts 37, 25, and 2. ∎

Fact 40 (Successor) Let $x, y \in \mathscr{Z}$ and $x \subset y$. Then $\mathscr{P}x \subseteq y$.

Proof Follows with Facts 23, 37 and 39. ∎

Theorem 41 (Completeness)
For every well-founded set x there exists a cumulative set y such that $x \in y$.

Proof Let $x \in \mathscr{W}$. We prove by induction on $x \in \mathscr{W}$ that there exists $y \in \mathscr{Z}$ such that $x \in y$. Let

$$u := \bigcup((\lambda y. L^{\mathscr{Z}}\{z \in \mathscr{Z} \mid y \in z\})@x)$$

It is easy to see that $u \in \mathscr{Z}$. Since $u \cup \mathscr{P}u \in \mathscr{Z}$, it suffices to show that $x \subseteq u$. Let $y \in x$. It suffices to show that there is some $v \in L^{\mathscr{Z}}\{z \in \mathscr{Z} \mid y \in z\}$ such that $y \in v$. By the inductive hypothesis we know that $\{z \in \mathscr{Z} \mid y \in z\}$ is inhabited. The claim follows since $I_{\mathscr{Z}}$ is a well-ordering. ∎

Fact 42 $I_{\mathscr{O}}$ and $I_{\mathscr{Z}}$ are isomorphic.

Proof By Theorems 31 and 38 and Fact 19 we know that $I_{\mathscr{O}}$ and $I_{\mathscr{Z}}$ are complete well-orderings. Thus $I_{\mathscr{O}}$ and $I_{\mathscr{Z}}$ are isomorphic by Fact 17. ∎

Fact 43 (Completeness) Let R be a realizable well-ordering on sets. Then there exists a unique $x \in \mathscr{Z}$ such that R and $I_{\mathscr{Z}}|\Sigma_x^{\mathscr{Z}}$ are isomorphic.

Proof Follows with Theorem 38 and Fact 17. ∎

Fact 44 The class $\mathscr{Z}'$ of sets inductively defined by the following rules contains exactly the cumulative sets.

$$\frac{x \subseteq \mathscr{Z}'}{\bigcup x \in \mathscr{Z}'} \qquad\qquad \frac{x \in \mathscr{Z}'}{\mathscr{P}x \in \mathscr{Z}'}$$

Proof Follows from the fact that $x \subseteq \mathscr{P}x$ for every transitive set x. ∎

The next fact asserts that our definition of the cumulative hierarchy agrees with the standard definition, which obtains the cumulative sets by transfinite recursion on ordinals [5, 7].

Fact 45 (Recursive Characterization) Let V be an isomorphism from $\mathscr{O}$ to $\mathscr{Z}$. Then:

1. If Vxz, then $V(x \cup \{x\})(\mathscr{P}z)$.
2. If $x \subseteq \mathscr{O}$, then $V(\bigcup x)(\bigcup(V@x))$.

Proof Claim 1 follows with Facts 32 and 40. For Claim 2, assume $x \subseteq \mathscr{O}$. Then $\bigcup x \in \mathscr{O}$. Let $V(\bigcup x)u$. It suffices to show $\bigcup(V@x) \subseteq u$ and $\bigcup(V@x) \not\subset u$.

We show $\bigcup(V@x) \subseteq u$. Let $y \in x$. Then $y \subseteq \bigcup x$. Moreover, $y \in \mathscr{O}$ since $\mathscr{O}$ is transitive. Let Vyz. We have $z \subseteq u$. The claim follows.

Suppose $\bigcup(V@x) \subset u$. Since $\bigcup(V@x) \in \mathscr{Z}$, we have $y \in \mathscr{O}$ such that $Vy(\bigcup(V@x))$ and $y \subset \bigcup x$. Since $\mathscr{O}$ is cumulative, we have $y \in \bigcup x$. Hence $y \in a \in x$ for some a. Since $a \in \mathscr{O}$, we have Vab for some $b \in \mathscr{Z}$. We have $\bigcup(V@x) \subset b$ since $y \subset a$ since $\mathscr{O}$ is cumulative. Contradiction since $b \in V@x$. ∎

Theorem 46 (Explicit Characterization) Let V be an isomorphism from $\mathscr{O}$ to $\mathscr{Z}$. Then Vxz if and only if $x \in \mathscr{O}$ and z is the least $z \in \mathscr{Z}$ such that $x \subseteq z$.

Proof Let $V := \lambda xz.\ x \in \mathscr{O} \wedge z \in L^{\mathscr{Z}}\{z \in \mathscr{Z} \mid x \subseteq z\}$. By Fact 14 it suffices to show that V is a maximal similarity from $\mathscr{O}$ to $\mathscr{Z}$. A proof of this fact can be found in the Coq development accompanying the paper. ∎

12 Further Characterizations of Ordinals

We show that our definition of ordinals is equivalent with Robinson's [11] definition and the characterization of ordinals as well-founded hereditarily transitive sets.

We define a class $\mathscr{O}'$ of sets inductively with a single rule.

$$\frac{x \subseteq \mathscr{O}' \qquad x \text{ transitive}}{x \in \mathscr{O}'}$$

We see $\mathscr{O}'$ as the class of well-founded hereditarily transitive sets. We will show that $\mathscr{O} \equiv \mathscr{O}'$.

Lemma 47 $\mathscr{O}' \subseteq \mathscr{W}$.

Lemma 48 Let $x \in \mathscr{O}'$. Then $\bigcup x \in \mathscr{O}'$ and $x \cup \{x\} \in \mathscr{O}'$.

Lemma 49 Let $x, y \in \mathscr{O}'$. Then either $x \in y$ or $y = x$ or $y \in x$.

Proof By nested induction on $x \in \mathscr{O}'$ and $y \in \mathscr{O}'$. Let $x \notin y$ and $y \notin x$. We show $x = y$ using extensionality.

Let $z \in x$. We show $z \in y$. By the inductive hypothesis for x we have either $z \in y$ or $z = y$ or $y \in z$. The first case is trivial. The second case is contradictory. The third case is also contradictory since x is transitive.

The other direction $y \subseteq x$ follows analogously. ∎

Lemma 50 Let $x \in \mathcal{O}'$. Then either $x = \bigcup x$ or $x = \bigcup x \cup \{\bigcup x\}$.

Proof We have $x \subseteq \mathcal{O}'$ and x transitive by inversion. We also have $\bigcup x \in \mathcal{O}'$ by Lemma 48. We apply Lemma 49 to x and $\bigcup x$ and have either $x \in \bigcup x$ or $x = \bigcup x$ or $\bigcup x \in x$. The first case is contradictory since $\bigcup x \subseteq x$ since x is transitive and $x \notin x$ since x is well-founded by Lemma 47. The second case is trivial. For the third case, let $\bigcup x \in x$. It suffices to show that $x \subseteq \bigcup x \cup \{\bigcup x\}$. Let $y \in x$. We have $y \in \mathcal{O}'$ since $x \subseteq \mathcal{O}'$. We apply Lemma 49 to y and $\bigcup x$ and have either $y \in \bigcup x$ or $y = \bigcup x$ or $\bigcup x \in y$. The first and the second case are trivial. The third case is contradictory since $y \subseteq \bigcup x$ since $y \in x$ and $\bigcup x$ is well-founded by Lemma 47. ∎

Lemma 51 $\mathcal{O}' \subseteq \mathcal{O}$.

Proof Let $x \in \mathcal{O}'$. We prove $x \in \mathcal{O}$ by induction on $x \in \mathcal{O}'$. By the inductive hypothesis we have $x \subseteq \mathcal{O}$. By Lemma 50 we have either $x = \bigcup x$ or $x = \bigcup x \cup \{\bigcup x\}$. In both cases the claim follows with Lemma 48. ∎

Theorem 52 (Characterizations) The following statements are equivalent.

1. $x \in \mathcal{O}$.
2. $x \in \mathcal{O}'$.
3. $x \in \mathcal{W}$ and $x \in \mathcal{T}$ and $x \subseteq \mathcal{T}$.
4. $x \in \mathcal{W}$ and $x \in \mathcal{T}$ and for all $y, z \in x$ either $y \in z$ or $y = z$ or $z \in y$.
5. $x \in \mathcal{T}$ and $(\lambda yz.\, y \in z \vee y = z)|x$ is a well-ordering.

Proof We show $(1) \to (4) \to (3) \to (2) \to (1)$. Verification of the equivalence of (4) and (5) is not difficult.

$(1) \to (4)$. Let $x \in O$. By Theorem 31 we know that x is well-founded and transitive. Let $y, z \in x$. Then $y, z \in \mathcal{O}$ by Fact 33. Hence $y \in z$ or $y = z$ or $z \in y$ by Theorem 31.

$(4) \to (3)$. Straightforward.

$(3) \to (2)$. By induction on $x \in \mathcal{W}$.

$(2) \to (1)$. By Lemma 51. ∎

Statement 4 of Theorem 52 expresses Robinson's definition [11] of ordinals. Since Robinson [11] assumes ZF with regularity, he can drop the well-foundedness requirement. Statement 3 characterizes ordinals as well-founded hereditarily transitive sets. Shoenfield [12] defines ordinals as hereditarily transitive sets; he does not require well-foundedness since he assumes regularity. Hrbacek and Jech [7] and others define ordinals using Statement 5 of Theorem 52. This characterization has the advantage that it does not require regularity and is easily expressed in a first-order setting.

References

[1] Wilhelm Ackermann. Die Widerspruchsfreiheit der allgemeinen Mengenlehre. *Mathematische Annalen*, 114(1):305–315, 1937.

[2] Peter Aczel. *Non-well-founded sets*. CSLI Lecture Notes 14. Stanford University, 1988.

[3] Peter Aczel. On relating type theories and set theories. In *Proc. of Types for Proofs and Programs, TYPES '98, LNCS 1657*, pages 1–18. Springer, 1998.

[4] Bruno Barras. Sets in Coq, Coq in sets. *J. Formalized Reasoning*, 3(1):29–48, 2010.

[5] Herbert B. Enderton. *Elements of Set Theory*. Academic Press, 1977.

[6] Thomas Forster. *Logic, Induction and Sets*. Cambridge University Press, 2003.

[7] Karel Hrbacek and Thomas Jech. *Introduction to Set Theory, Third Edition, Revised and Expanded*. CRC Press, 1999.

[8] Zhaohui Luo. *Computation and Reasoning: A Type Theory for Computer Science*. Oxford University Press, 1994.

[9] John von Neumann. Zur Einführung der transfiniten Zahlen. *Acta Scientiarum Mathematicarum (Szeged)*, 1(4):199–208, 1923. English translation in van Heijenoort.

[10] John von Neumann. Über die Definition durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre. *Mathematische Annalen*, 99(1):373–391, 1928.

[11] Raphael M. Robinson. The theory of classes, a modification of von Neumann's system. *The Journal of Symbolic Logic*, 2(1):29–36, 1937.

[12] Joseph R. Shoenfield. *Mathematical logic*. Addison-Wesley, 1967.

[13] Gert Smolka, Steven Schäfer, and Christian Doczkal. Transfinite constructions in classical type theory. Submitted for publication, Mach 2015.

[14] Patrick Suppes. *Axiomatic Set Theory*. Van Nostrand, 1960.

[15] Benjamin Werner. Sets in types, types in sets. In *Proc. of Theoretical Aspects of Computer Software, LNCS 1281*, pages 530–346. Springer, 1997.