Set Theory in Type Theory

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How would you teach set theory to students who are familiar with type theory and proof assistants?

- Classical set theory with Zermelo-Fraenkel axioms
- Type theory with XM and impredicative Prop
- Coq as proof assistant
- Perspective very different from mathematical textbooks
- Explore an axiomatization in an expressive, explicit, and familiar logic
Axioms

\[ S : \text{Type} \]
\[ \in : S \to S \to \text{Prop} \]

\[ x = y \iff x \equiv y \]

\[ z \in \emptyset \iff \bot \]
\[ z \in \{x, y\} \iff z = x \lor z = y \]
\[ z \in \bigcup x \iff \exists y \in x. \ z \in y \]
\[ z \in \mathcal{P}x \iff z \subseteq x \]
\[ z \in R@x \iff \exists y \in x. \ Ryz \land \text{unique (Ry)} \]

- Replacement axiom is higher-order, \( R : S \to S \to \text{Prop} \)
- Infinity and choice are not needed for this talk
A **class** is a predicate $p : S \rightarrow \text{Prop}$

Not every class can be represented as a set, e.g., $\lambda x . x \notin x$

Type theory provides classes and relations on classes

Classes are not formalized by Zermelo-Fraenkel set theory

Von-Neumann-Gödel-Bernays set theory accommodates sets and classes in first-order logic
Separation and Description

Separation

\[ z \in x \cap p \iff z \in x \land px \]

Description

\[ p \vdash p \downarrow \iff p \text{ unique and inhabited} \]

An operator that maps relations \( R \) on \( S \) to total functions \( f : S \to S \) such that \( f \) agrees with \( R \) on unique images can be expressed (i.e., \( Rx(fx) \))
Numbers and Ordered Pairs

can be represented as sets

- Functions, numbers, and pairs already exist in type theory
- Can express functions \( \bar{\cdot} : \mathbb{N} \rightarrow \mathbb{S} \), \( \text{succ} : \mathbb{S} \rightarrow \mathbb{S} \), and \( \text{pred} : \mathbb{S} \rightarrow \mathbb{S} \) such that:

  \[
  \bar{m} = \bar{m} \iff m = n \\
  \text{succ} \bar{n} = \bar{n + 1} \\
  \text{pred} \bar{n + 1} = \bar{n}
  \]

- Can express functions \( \text{pair} : \mathbb{S} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \), \( \text{fst} : \mathbb{S} \rightarrow \mathbb{S} \), and \( \text{snd} : \mathbb{S} \rightarrow \mathbb{S} \) such that:

  \[
  \text{pair} x y = \text{pair} x' y' \rightarrow x = x' \land y = y' \\
  \text{fst} (\text{pair} x y) = x \\
  \text{snd} (\text{pair} x y) = y
  \]

- [Barras 2010] [von Neumann 1923] [Kuratowski 1921]
Can Construct Models of Axioms

- Without infinity hereditarily finite sets suffice
- Use Ackermann encoding into numbers
- Need strong excluded middle for replacement ($\text{Prop} \simeq \text{bool}$)
- Aczel, Werner, Miquel construct models with infinite sets
Cumulative Hierarchy

- Horizontal lines represent stages (successors and limits)
- Blue lines also represent slices
- Every well-founded set appears in some slice
- Stages are well-ordered
- Every well-ordered set is order-isomorphic to a unique segment
Define class $\mathcal{W}$ of **well-founded sets** inductively

$$
\begin{align*}
x \subseteq \mathcal{W} & \implies x \in \mathcal{W}
\end{align*}
$$

- Well-founded sets are defined as sets that admit $\epsilon$-induction
- Inductive definition unknown in set theory
- Regularity axiom can be expressed as $\forall x. x \in \mathcal{W}$
- First-order characterization of $x \in \mathcal{W}$ seems to require infinity (to express transitive closure)
- First-order characterization of $x \in \mathcal{W} \cap T$ straightforward
- Aczel [1988] studies non-well-founded sets
- $\mathcal{W}$ cannot be represented as a set
Stages of Cumulative Hierarchy

- Define class $\mathcal{L}$ of **cumulative sets** inductively

\[
\begin{align*}
  &x \subseteq \mathcal{L} \\
  \implies & \bigcup x \in \mathcal{L} \\
  &x \in \mathcal{L} \\
  \implies & x \cup \mathcal{P}x \in \mathcal{L}
\end{align*}
\]

- $\mathcal{L}$ well-ordered by $\subseteq$, unbounded, $\emptyset$ least element
- $W \equiv \bigcup \mathcal{L}$
- $x \subset y$ iff $x \in y$ for all $x, y \in \mathcal{L}$
- $x \cup \mathcal{P}x = \mathcal{P}x$ if $x \in \mathcal{L}$ since $\mathcal{L} \subseteq \mathcal{I}$
- Definition of $\mathcal{L}$ is instance of tower construction
- $\mathcal{L}$ usually defined with transfinite induction on ordinals
Ordinals

- Define class $\mathcal{O}$ of ordinals inductively

\[ x \subseteq \mathcal{O} \implies \bigcup x \in \mathcal{O} \]
\[ x \in \mathcal{O} \implies x \cup \{x\} \in \mathcal{O} \]

- Every cumulative slice contains exactly one ordinal
- Every ordinal is the set of all smaller ordinals
- Every well-ordered set is order isomorphic to a unique ordinal
- $\mathcal{O}$ order isomorphic with $\mathbb{I}$
- Definition of $\mathcal{O}$ is instance of tower construction
First-Order Characterization of Ordinals

- Ordinals are hereditarily transitive and well-founded sets [Bernays 1931]

- $x \in \mathcal{O}$ iff $x \in \mathcal{T}$ and $x \subseteq \mathcal{T}$ and $x \in \mathcal{W}$

- $x \in \mathcal{O}$ iff $x \in \mathcal{T}$ and $x \subseteq \mathcal{T}$ and $\mathcal{P}x \subseteq \mathcal{R}$

- $\mathcal{T} := \{ x \mid \forall y \in x. \ y \subseteq x \}$ transtive sets

- $\mathcal{R} := \{ x \mid \exists y \in x. \ \forall z \in x. \ z \notin y \}$ regular sets

- If $x \in \mathcal{T}$, then $x \in \mathcal{W}$ iff $\mathcal{P}x \subseteq \mathcal{R}$

- Corresponding inductive characterization:

  \[
  \begin{align*}
  x \in \mathcal{T} & \quad x \subseteq \mathcal{O} \\
  \hline
  x \in \mathcal{O}
  \end{align*}
  \]
Tower Construction for Sets

- Assume $f : S \to S$
- Define class $T$ of sets inductively:

\[
\begin{align*}
  x & \subseteq T \\
  \bigcup x & \in T \\
  x & \in T \\
  x \cup f x & \in T
\end{align*}
\]

- $T$ is well-ordered by $\subseteq$, $\emptyset$ least element
- $x \cup f x$ successor of $x$ if $x \in T$ not maximal
- Every segment of $T$ can be represented as a set
- If $f$ preserves transitivity and well-foundedness, and $x \in f x$ for all $x$,
  - $T$ unbounded
  - $T$ cannot be represented as a set
  - Every well-ordered set is isomorphic to a proper segment of $T$
- $x \in y$ iff $x \subset y$ for all $x, y \in T$
Assume type $X$ and partial order $\leq$

Assume $x_0 : X$

Assume increasing function $f : X \to X$ (i.e., $x \leq f x$)

Assume family $\mathcal{I}$ of classes on $X$, closed under subclasses

Assume function $\bigsqcup$ that yields supremum for every $p \in \mathcal{I}$

Define class $T$ on $X$ inductively:

$\begin{align*}
& x_0 \in T \\
& x \in T \quad f x \in T \\
& p \subseteq T \\
& p \in \mathcal{I} \\
& p \text{ inhabited}
\end{align*}$

$\bigsqcup p \in T$

$T$ well-ordered by $\leq$ ($x_0$ least element, $f$ yields successors)

$T$ unbounded iff $f$ has no fixed point in $T$

If $T \in \mathcal{I}$, then $\bigsqcup T$ is unique fixed point of $f$ in $T$

(Bourbaki-Witt theorem)

See forthcoming paper at ITP 2015
Final Remarks

- Type theory provides expressive language for talking about sets and classes
  - more natural than first-order logic
  - first-order encodings are low-level and tedious; e.g.,
    - well-founded sets
    - von-Neumann-Gödel-Bernays set theory
- Many aspects of set theory can be formulated more generally at the level of type theory:
  - Well-orderings
  - Transfinite recursion
  - Tower construction
  - Well-ordering theorem
- Cumulative hierarchy can be considered before ordinals, transfinite recursion is not needed