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MASTER'S THESIS

STRUCTURAL ASPECTS OF BANACH CATEGORIES

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Abstract

Probabilistic programming languages (PPLs) are a useful tool for statistical modelling, and semantics of PPLs are a highly active field of research. In the literature, two main approaches to give semantics to PPLs have emerged: Semantics based on categories of linear operators and semantics based on categories of Markov kernels, where a Markov kernel is a generalisation of a stochastic matrix. Both approaches have its strengths and weaknesses, and its weaknesses are complementary. Addressing these weaknesses is, traditionally, both highly technical and subtle.

Recent advance has been achieved by Azevedo de Amorim who defines a two-level calculus nicely combining both types of PPLs: One level can be interpreted by categories of linear operators and the other one using categories of Markov kernels, while a modality mediates between both levels, which is interpreted by a lax monoidal functor. We introduce the term **Banach category** to refer to models of this calculus.

Azevedo de Amorim's PPL has already been applied in the literature, but the theory of Banach categories remains underdeveloped, an issue we address in this thesis. Using string diagrams, a well-known calculus to reason about monoidal categories, we investigate how the structure of Markov kernels and the lax monoidal functor interact. Azevedo de Amorim further points out that the semantics of his calculus behave even better when the lax monoidal functor is full, motivating us to prove our main theorem: Subject to certain conditions, for any Banach category, one can find another Banach category – its fullification – such that the lax monoidal functor of the fullification is full and the original Banach category and its fullification are related in a natural way.

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Contents

Chapter 1

Introduction

In this master's thesis, we investigate **Banach Categories**, which is a recent, so far unnamed, concept in category theory in the subfield of categorical probability theory. We contribute to a model proposed by Azevedo de Amorim [15], which extends Markov categories [5, 27], a frequently applied abstraction of Markov kernels, with notions of linear operators, combining two previously distinct directions in probabilistic semantics [10, 12, 21, 18, 41, 52] into a single entity. We are particularly interested in questions concerning the connections of Azevedo de Amorim's model and theoretical computer science as well as categorical logic. We are motivated by the fact that Azevedo de Amorim's abstraction captures useful models and has first applications in the literature, e.g. by him, Witzman, and Kozen [16], but the theory of his abstraction is still underdeveloped.

1.1 Probabilistic Programming Languages and Categorical Semantics

Azevedo de Amorim's [15] work arises in the scope of *probabilistic programming languages* (PPLs) [11], which, in its broader sense, are programming languages with probabilistic primitives. PPLs are used in statistical modelling, for instance, to account for the involved randomness. Semantics to PPLs can be given via categories, and different kinds of PPLs are interpreted by different kinds of categories. Today, there are two key classes of PPLs, which are explained below.

Linear languages. On the one hand, there are PPLs based on Girard's [28] linear logic. These languages are higher-order, but have the restriction that variables can be used only once, i.e. in a linear way. In many models of linear languages, variables of ground type range over distributions, and using a variable means sampling from the distribution modelled by the variable. The linearity constraint then bakes in the fact that sampling is a non-repeatable process.

Usually, models for linear languages are certain vector spaces and linear functions between them. One notable example is due to Dahlqvist and Kozen [10], who define a higher-order linear probabilistic language and give semantics to it using certain Banach spaces. Also Ehrhard and colleagues [12, 21, 18] have made significant contributions to linear languages. A key drawback of linear languages is that it is hard to program with them due to the linearity restriction. There are ways to accommodate non-linear programs via Girard's exponential modality [28], often written "!", which are used both by Ehrhard et al. as well as Dahlqvist and Kozen. A particularly elegant solution has been proposed by Ehrhard and colleagues. First in the discrete case [20, 21], but it has recently been extended to the continuous case as well [19].

Markov kernel languages. On the other hand, there are PPLs based on Markov kernels, avoiding these issues by design: Variables of ground type can be thought of ranging over values, not distributions. A Markov kernel is a generalisation of a stochastic matrix to the continuous case. The key difference to the linear languages is that variables can be copied and therefore used several times. A major drawback, however, is that it is highly difficult to add higher-order functionality to Markov kernel languages. In fact, this has only been achieved rather recently with the introduction of quasi-Borel spaces by Heunen, Kammar, Staton, and Yang [31]. Still, results from a linear algebraic treatment of probability theory, a more feasible foundation for the linear case, do not apply to quasi-Borel spaces.

Markov kernel Languages date back to the 1980s with work by Kozen [41]. Panangaden [52] observed that categories of Markov kernels are suitable models for these languages. In particular, Kleisli categories [40] of probability monads, for instance the Giry monad [29], are paradigmatic examples of models for Markov kernel languages. Azevedo de Amorim [15] gives a modern, functional Markov kernel language and shows that his approach is compatible with traditional linear algebraic models of linear logic such as categories of Banach spaces.

Combining both directions as single entity. In general, models of both types of languages are certain monoidal categories [3, 47], which are categories with a weak version of products, more precisely products that may not necessarily have projections. Models of linear languages are *symmetric monoidal closed categories* [22] (SMMCs) which can natively express higher-order functionality, and models of Markov kernel languages are *Markov categories* [5, 27], introduced in the late 2010s, which have built-in support for copying.

Azevedo de Amorim [15] introduces a two-level PPL with one level being interpreted by an SMCC and the other level being interpreted by a Markov category. Variables of ground type in the higher-order SMCC level range over distributions and cannot be copied, while variables in the first-order Markov level range over values which can be copied freely. The translation between both levels can be seen as sampling: A distribution, living in the SMCC level, can be sampled from, and the resulting value, which lives in the Markov level, can then be used freely.

The translation between the levels is achieved via a *lax monoidal functor* from the Markov category to the SMCC. Roughly speaking, this is a mapping between the categories that preserves the monoidal structure to a sufficient extent. Models of Azevedo de Amorim's language are therefore triples: A Markov category, an SMCC, and a lax monoidal functor. In this thesis, we introduce the term *Banach category* for these triples.

1.2 Diagrammatic Reasoning with String Diagrams

In monoidal categories, many concepts, such as associativity of the monoidal product, are defined *up to isomorphism*. While this gives rise to a wide range of examples of monoidal categories, formal reasoning about morphisms in such categories becomes extremely tedious as the isomorphisms of the monoidal structure occur ubiquitously. To alleviate these difficulties, *string diagrams* [53, 7] are used for diagrammatic reasoning in monoidal categories. This is a graphical calculus operating on equivalence classes of morphisms modulo the isomorphisms of the monoidal structure, which simplifies reasoning greatly. There are also graphical gadgets to incorporate additional structure, such as copying in Markov categories.

1.3 Historical Remarks

Category theory. The scene for this thesis is set in the scope of category theory, which originated in the late 1940s jointly by Eilenberg and Mac Lane [24] as a tool to better understand algebraic topology and homology theory. Remarkably, almost all of the fundamental categorical definitions are introduced in this single paper, yet some examples of the new categorical definitions implicitly occurred a few years prior [23].

Categorical semantics. Since the 1940s, category theory has come a long way and is by now a versatile tool not only applied in algebra, but also theoretical computer science. Prominently, category theory has vast applications in logic and the theory of programming languages, for instance via categorical semantics, where programs are interpreted as morphisms in a category. A key observation, made formally precise by Lambek and Scott [43], is that certain categories, called Cartesian closed categories [44], can be used to give semantics to Church's simply typed λ-calculus [6], *the* paradigmatic theoretical underpinning of functional programming. Also, Lambek and Scott make precise that Cartesian closed categories are equivalent to intuitionistic logic. These equivalences are the result of work in the 1970s, mostly due to Lambek, and they are the categorical counterpart to the celebrated Curry-Howard correspondence [9, 32] between simply typed λ-calculus and intuitionistic logic. The equivalence between all three is sometimes referred to as *Curry-Howard-Lambek correspondence*.

In the same spirit, SMCCs and linear languages as well as Markov categories and Markov kernel languages show the same equivalence, which motivates their categorical study explained earlier.

1.4 Contributions

Azevedo de Amorim's [15] calculus behaves even better when the lax monoidal functor of the Banach category is full: Fullness would give a way to translate from the SMCC level back to the Markov level in certain circumstances. This is particularly useful since under the assumption of fullness, functions in the SMCC level between two types induced from the Markov level are themselves only Markov kernels, and can hence be programmed with the freedom of copying variables. In the present thesis, we investigate this fullness condition and give a construction to turn lax monoidal functors into full functors (i.e. we "fullify" them). Concretely, the present thesis contributes the following:

- We formally introduce the name **Banach category** for a triple of a Markov category $\mathcal{C}$, an SMCC $\mathcal{L}$, and a lax monoidal functor $\mathsf{F} : \mathcal{C} \to \mathcal{L}$ (Definition 2.27), following Azevedo de Amorim's [15] observation that these triples are well-suited models to combine linear languages and Markov kernel languages.
- We investigate, in terms of string diagrams, how the lax monoidal functor of a Banach category interacts with the structure of the Markov category (Section 3.2). Further, we point out that strong monoidal functors have even better graphical properties.
- We prove that for any Banach category $(\mathcal{C}, \mathsf{F}, \mathcal{L})$ where $\mathcal{C}$ is small, there exists a Banach category $(\mathcal{C}, \widetilde{\mathsf{F}}, \widetilde{\mathcal{L}})$ (called the **fullification** of $(\mathcal{C}, \mathsf{F}, \mathcal{L})$) and a strict monoidal functor $\mathsf{q} : \widetilde{\mathcal{L}} \to \mathcal{L}$ such that $\widetilde{\mathsf{F}}$ is full and the following diagram commutes (Corollary 4.28):

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{\mathsf{F}}"] \arrow[dr, "\mathsf{F}"'] & \widetilde{\mathcal{L}} \arrow[d, "\mathsf{q}"] \\ & \mathcal{L} \end{tikzcd}$$

- We demonstrate that the mapping $(\mathcal{C}, \mathsf{F}, \mathcal{L}) \mapsto (\mathcal{C}, \widetilde{\mathsf{F}}, \widetilde{\mathcal{L}})$ is a functor (Proposition 4.30), which gives rise to future directions to study this fullification functor, for instance with respect to universal properties via adjunctions.

1.5 Outline

First, we give an overview of the definitions and results of category theory used in this thesis in Chapter 2, which includes the definition of Banach categories. Then, in Chapter 3, we investigate the interplay of the lax monoidal functor and the structure of the Markov category in a Banach category graphically. This is motivated by a brief example of convex spaces, discussed in Section 3.1. After this, we proceed to the main theorem of this thesis in Chapter 4. We first review a few technical preliminaries, and then prove our main result. This is split into two parts: First, we only define the full functor in Section 4.5, and then proceed to verify that the fullification is indeed a Banach category in Section 4.6. The thesis is rounded up by a conclusion and discussion of related work in Chapter 5.

Chapter 2

Background in Category Theory

All results and arguments presented in this thesis are formulated in the language of category theory [24]. This chapter provides the required background in category theory to follow all arguments presented in the present thesis.

First, the notion of a category and all needed fundamentals from a first course in category theory are given (Section 2.1); all these definitions are standard. Then, a brief overview over string diagrams [53, 7, 50] is given, which are a graphical tool to reason about morphisms in monoidal categories [3, 47] (Section 2.2). After that, we introduce the rather recent notion of Markov categories [27, 5], which are certain monoidal categories designed to provide a categorical approach to probability theory (Section 2.3). Finally, we define the Banach category consisting of monoidal functors from Markov categories to symmetric monoidal closed categories (Section 2.4).

The reader with a working knowledge in category theory may wish to skip Section 2.1, and perhaps even Section 2.2 if they have a solid knowledge of monoidal categories. Section 2.3 is a recommended read, and Section 2.4 is essential.

2.1 Fundamental Categorical Notions

The standard introductory textbook to category theory is Mac Lane's "Categories for the Working Mathematician" [48], which contains all the concepts introduced in this section. For the following definitions, we also borrowed the slightly more modern notation from Borceux [2, Chapters 1-3].

Categories. The first notion is the one of a **category**. Think of a mathematical structure, such as groups, fields, or vector spaces over a fixed field. They all come with objects, and (homo)morphisms between them. In fact, many structures in mathematics and theoretical computer science are organised in this way.

Definition 2.1 (Category) *A* ***category*** $\mathcal{C}$ *consists of*

- *a class* $\mathsf{Ob}(\mathcal{C})$ *of* ***objects*** *of* $\mathcal{C}$*,*

- *for each* $A, B \in \mathsf{Ob}(\mathcal{C})$*, a class* $\mathcal{C}(A, B)$ *of* ***morphisms*** *or* ***arrows*** *from* A *to* B*,*
- *for each* $A, B, C \in \mathsf{Ob}(\mathcal{C})$*, a class term*

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

  for the ***composition*** *operation. For* $f \in \mathcal{C}(A, B)$ *and* $g \in \mathcal{C}(B, C)$*, the morphism* $g \circ f$ *is called the* ***composite*** *of* f *and* g*, and*
- *for each* $A \in \mathsf{Ob}(\mathcal{C})$*, a morphism* $\mathrm{id}_A \in \mathcal{C}(A, A)$ *called the* ***identity*** *on* A*.*

This data is required to satisfy the following properties:

- *Associativity: For all objects* $A, B, C, D \in \mathsf{Ob}(\mathcal{C})$ *as well as* $f \in \mathcal{C}(A, B)$*,* $g \in \mathcal{C}(B, C)$*, and* $h \in \mathcal{C}(C, D)$*, it needs to hold that*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

- *Identity: For all objects* $A, B \in \mathsf{Ob}(\mathcal{C})$ *and* $f \in \mathcal{C}(A, B)$*, it needs to hold that*

$$f = \mathrm{id}_B \circ f = f \circ \mathrm{id}_A .$$

Consider again the example of groups. If one picks the class of all groups, Grp, as objects and for all groups G, G', sets

$$\mathsf{Grp}(G, G') := \{f : G \to G' \mid f \text{ is group homomorphism}\},$$

then one obtains a category. Also, the class of all sets, denoted Set, with functions as morphisms, gives rise to a category. Not all categories are such that morphisms are actual functions, as it is for Grp and Set. Consider any partially ordered set $(X, \leqslant)$. This gives a rise to a category $\mathcal{D}$ where $\mathsf{Ob}(\mathcal{D}) := X$ and

$$\mathcal{D}(x, y) := \begin{cases} \emptyset & \text{if } x \nleqslant y \\ \{*\} & \text{if } x \leqslant y \end{cases}.$$

By virtue of transitivity and reflexivity of $\leqslant$, composition and identities in $\mathcal{D}$ are trivial to define. Moreover, the associativity and identity laws trivially hold.

Notation-wise, we will usually write $A \in \mathcal{C}$ instead of $A \in \mathsf{Ob}(\mathcal{C})$. Further, appealing to the intuition that morphisms generalise the concept of functions, we will write $f : A \to B$ instead of $f \in \mathcal{C}(A, B)$. Here, A is called the **domain** of f and B the **codomain** of f. Also, we say that $A \to B$ is the **type** of f.

A category $\mathcal{C}$ such that, for all $A, B \in \mathcal{C}$, the class $\mathcal{C}(A, B)$ is a set will be called **locally small**. If, in addition, $\mathsf{Ob}(\mathcal{C})$ is a set, then $\mathcal{C}$ will be called **small**. From now on, we assume that all categories appearing in this text are locally small.

A key tool in category theory are **commutative diagrams**. A diagram is a directed graph with objects of a fixed category as vertices and morphisms in this category as edges (multiple edges between the same two vertices are allowed). We introduce them by example. Let $\mathcal{C}$ be a category and $A, B, C, D \in \mathcal{C}$. Then, for morphisms f, g, h, k, the diagram below is said to be commutative if $g \circ f = k \circ h$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Note that the types of the morphisms can be inferred from the diagram. This example scales to arbitrary (finite) diagrams. A general finite diagram is said to be **commutative** if any two paths between two vertices describe the same morphism. There is a formalism (via so-called **cones**) for diagrams that even accounts for infinite diagrams. Cones is not needed for this thesis and the interested reader is referred to the textbooks, e.g. Borceux [2, Section 2.6].

There is also a categorical version of **isomorphisms**.

Definition 2.2 *Let $\mathcal{C}$ be a category and $A, B \in \mathcal{C}$. A morphism $f : A \to B$ is an **isomorphism**, or **iso**, if there exists $f^{-1} : B \to A$ such that both $\mathrm{id}_A = f^{-1} \circ f$ and $\mathrm{id}_B = f \circ f^{-1}$.*

Constructions in categories and universal properties. Consider again the category Set of sets and functions. For sets A and B, there exists a Cartesian product $A \times B$ whose elements are pairs (a, b) such that $a \in A$ and $b \in B$. There are standard projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$. Whenever, for some set C, there are functions $f : C \to A$ and $g : C \to B$, then there exists a unique function $\langle f, g\rangle : C \to A \times B$ such that $f = \pi_1 \circ \langle f, g\rangle$ and $g = \pi_2 \circ \langle f, g\rangle$. Here, $\langle f, g\rangle$ is given by $c \mapsto (f(c), g(c))$. This is in fact a purely categorical property.

Definition 2.3 (Product) *Let $\mathcal{C}$ be a category and $A, B \in \mathcal{C}$. A **product** of A and B, written $A \times B$, is an object of $\mathcal{C}$ with arrows $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ having the following property:*

For any $C \in \mathcal{C}$ with arrows $f : C \to A$ and $g : C \to B$, there exists a unique arrow $\langle f, g\rangle : C \to A \times B$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, "\langle f,g\rangle"'] & \\ A & & B \\ & A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

The definition of products involves a **universal property**, which is a key concept in category theory. A universal property is, as the name suggests, some universally quantified statement that asserts a unique existence of something for each element the quantifier ranges over. In the example of products, this quantifier just ranges over objects C together with morphisms $C \to A$ and $C \to B$. The unique existence is asserted for the arrow $\langle f, g\rangle$. Arrows obtained from universal properties are written as dashed arrows in diagrams.

There is a way to define arbitrary finite products, one simply uses induction on the definition provided above. The 1-ary product of A is just A with the identity as projection, and a 0-ary product is a **terminal object**:

Definition 2.4 (Terminal Object) *Let $\mathcal{C}$ be a category. An object $T \in \mathcal{C}$ is called **terminal** if for all $A \in \mathcal{C}$, there exists a unique arrow of type $A \to T$.*

Again, the definition above involves a universal property. In Set, any singleton set is terminal.

One last, slightly more technical, definition is needed, which follows the same pattern. It is used in Section 4.1.

Definition 2.5 (Pullback) *Let $\mathcal{C}$ be a category and $A, B, C \in \mathcal{C}$ with arrows $f : A \to C$ and $g : B \to C$. A **pullback** of f and g is an object $P \in \mathcal{C}$ with arrows $p_1 : P \to A$ and $p_2 : P \to B$ such that*

- *the following diagram commutes, and*

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

- *for all $Q \in \mathcal{C}$ with arrows $q_1 : Q \to A$ and $q_2 : Q \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique arrow $t : Q \to P$ such that the following diagram commutes:*

$$\begin{tikzcd} Q \arrow[dr, dashed, "t"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd}$$

In diagrams, the corner to the bottom right of P is used to indicate a pullback. The second point in the definition above is the universal property of the pullback. In Set, a pullback of $f : A \to C$ and $g : B \to C$ is given by $\{(a, b) \in A \times B \mid f(a) = g(b)\}$.

Note that a product, a terminal object, or a pullback need not be unique. For instance, both $\{0\}$ and $\{1\}$ are terminal in Set. However, any two terminal objects are isomorphic, similar for products and pullbacks.

Also, not all categories need to have products, terminal objects, or pullbacks. For instance, the partially ordered set $\{a, b, c\}$ such that $b \geqslant a$ and $c \geqslant a$ but b and c are incomparable, does not have a terminal object.

All these definitions are similar: One specifies a set of objects and morphisms between them, and then asserts that some universal object with universal arrows exists for this structure. Indeed, all three notions introduced in this section are examples of **limits**. See Borceux [2, Chapter 2] for more details.

Constructions on categories. A few operations on categories will be needed: The **product** of two categories, and the key concept of the **opposite** of a category. The former is standard, the latter more interesting.

Definition 2.6 (Product Category) *Let $\mathcal{C}, \mathcal{D}$ be categories. The category $\mathcal{C} \times \mathcal{D}$ is defined as follows:*

- *Objects of $\mathcal{C} \times \mathcal{D}$ are pairs (A, B) such that $A \in \mathcal{C}$ and $B \in \mathcal{D}$.*
- *A morphism $f : (A_1, B_1) \to (A_2, B_2)$ in $\mathcal{C} \times \mathcal{D}$ is a pair $f = (a, b)$ such that $a : A_1 \to A_2$ is a $\mathcal{C}$-morphism and $b : B_1 \to B_2$ is a $\mathcal{D}$-morphism.*
  - *The composition of $(a, b) : (A_1, B_1) \to (A_2, B_2)$ and $(a', b') : (A_2, B_2) \to (A_3, B_3)$ is defined as*

$$(a', b') \circ (a, b) := (a' \circ a, b' \circ b).$$

  - *The identity on (A, B) is given by $\mathrm{id}_{(A,B)} := (\mathrm{id}_A, \mathrm{id}_B)$.*

*The category $\mathcal{C} \times \mathcal{D}$ is called the **product category**, or just **product**, of $\mathcal{C}$ and $\mathcal{D}$.*

For product categories, associativity and identity laws are induced from $\mathcal{C}$ and $\mathcal{D}$, respectively. Slightly more interesting are opposite categories.

Definition 2.7 (Opposite Category) *Let $\mathcal{C}$ be a category. Define a category $\mathcal{C}^{\mathsf{op}}$ as follows:*

- $\mathsf{Ob}(\mathcal{C}^{\mathsf{op}}) := \mathsf{Ob}(\mathcal{C})$,
- *for $A, B \in \mathcal{C}^{\mathsf{op}}$, $\mathcal{C}^{\mathsf{op}}(A, B) := \mathcal{C}(B, A)$, and*
  - *composition as well as identities as in $\mathcal{C}$.*

*The category $\mathcal{C}^{\mathsf{op}}$ is the **opposite category**, or **dual**, of $\mathcal{C}$.*

The opposite of a given category $\mathcal{C}$ is just as $\mathcal{C}$, but with reversed arrows. The concept of duals will become useful when defining functors that reverse arrows, as in the Yoneda lemma (Theorem 2.15), for instance.

Functors. As with many mathematical structures, there exists a notion of "map" between two categories. This is the concept of a **functor**.

Definition 2.8 (Functor) *Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ consists of:*

- *a mapping $\mathsf{F} : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D})$. For $A \in \mathcal{C}$, the object obtained by applying F to A is denoted by $\mathsf{F}\,A$, and*
- *for each $A, B \in \mathsf{Ob}(\mathcal{C})$, a mapping $\mathsf{F} : \mathcal{C}(A, B) \to \mathcal{D}(\mathsf{F}\,A, \mathsf{F}\,B)$. For $f : A \to B$, the morphism obtained by applying F to f is denoted by $\mathsf{F}\,f$.*

This data is required to satisfy the following properties:

- *For all objects $A, B, C \in \mathcal{C}$ as well as $f : A \to B$ and $g : B \to C$, it needs to hold that*
$$\mathsf{F}\,(g \circ f) = \mathsf{F}\,g \circ \mathsf{F}\,f.$$
- *For all objects $A \in \mathcal{C}$, it needs to hold that*
$$\mathsf{F}\ \mathrm{id}_A = \mathrm{id}_{\mathsf{F}\,A}\,.$$

*F is said to be **full** (**faithful**) if each mapping $\mathsf{F} : \mathcal{C}(A, B) \to \mathcal{D}(\mathsf{F}\,A, \mathsf{F}\,B)$ is surjective (injective). If F is both full and faithful, we say that F is **fully faithful**. Further, if F is faithful and injective on objects, we say that F is an **embedding**.*

There are many natural examples for functors. One important class of functors are **forgetful** ones: There is a functor $\mathsf{U} : \mathsf{Grp} \to \mathsf{Set}$ which sends a group to its underlying set and a group homomorphism to the underlying function it represents. That is, U "forgets" the group structure. Also, consider two partially ordered sets $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$, seen as a category. A functor $\mathsf{F} : (X, \leqslant_X) \to (Y, \leqslant_Y)$ is a function $X \to Y$ such that for $x_1, x_2 \in X$, it holds that $x_1 \leqslant_X x_2$ implies $\mathsf{F}\,x_1 \leqslant_Y \mathsf{F}\,x_2$.

An important class of functors are so-called **hom-functors**.

Definition 2.9 (Covariant Hom-Functor) *Let $\mathcal{C}$ be a category and $C \in \mathcal{C}$. The **covariant hom-functor** $\mathcal{C}(C, -) : \mathcal{C} \to \mathsf{Set}$ is defined as follows:*

- *On objects, define $\mathcal{C}(C, -)\,D := \mathcal{C}(C, D)$.*
- *On arrows $f : D \to E$, define $\mathcal{C}(C, -)\,f := f \circ -$, where "$f \circ -$" is a function $\mathcal{C}(C, D) \to \mathcal{C}(C, E)$ such that $h \mapsto f \circ h$.*

Definition 2.10 (Contravariant Hom-Functor) *Let $\mathcal{C}$ be a category and $C \in \mathcal{C}$. The* ***contravariant hom-functor*** $\mathcal{C}(-, C) : \mathcal{C}^{op} \to \mathsf{Set}$ *is defined as follows:*

- *On objects, define* $\mathcal{C}(-, C)\, D := \mathcal{C}(D, C)$.
- *On arrows* $f : D \to E$ (*i.e.* f *is a morphism of type* $E \to D$ *in* $\mathcal{C}$), *define* $\mathcal{C}(-, C)\, f := - \circ f$, *where "*$- \circ f$*" is a function* $\mathcal{C}(D, C) \to \mathcal{C}(E, C)$ *such that* $h \mapsto h \circ f$.

The operation denoted by $f \circ -$ is called **postcomposition**, and the operation $- \circ f$ is called **precomposition**.

Hom-functors play a key role in the Yoneda lemma (Theorem 2.15), a key result which is used to define the fullification of a lax monoidal functor (Corollary 4.17).

Yet another important example of a functor is the tensor product of vector spaces. If $\mathsf{Vect}_\mathbb{R}$ denotes the category of real vector spaces with linear maps between them, then the tensor product $\otimes : \mathsf{Vect}_\mathbb{R} \times \mathsf{Vect}_\mathbb{R} \to \mathsf{Vect}_\mathbb{R}$ is indeed a functor. Here, $\mathsf{Vect}_\mathbb{R} \times \mathsf{Vect}_\mathbb{R}$ is the product category of $\mathsf{Vect}_\mathbb{R}$ with itself. This example reappears in Section 2.2.

With functors, we can succinctly define the notion of a **subcategory**. A subcategory is to a category what a subset is to a set.

Definition 2.11 (Subcategory) *Let $\mathcal{C}, \mathcal{D}$ be categories. We say that $\mathcal{C}$ is a* ***subcategory*** *of $\mathcal{D}$ if there is an embedding $\mathcal{C} \to \mathcal{D}$. Further, $\mathcal{C}$ is a* ***full subcategory*** *if there is an embedding $\mathcal{C} \to \mathcal{D}$ that is full.*

Categories and functors form a category as well, up to size issues. We will therefore also draw commutative diagrams where categories are vertices and functors are edges. For the technical details see e.g. Borceux [2].

Natural transformations and Yoneda lemma. There is a notion of morphism between functors, called **natural transformations**. In fact, natural transformations are the reason why Eilenberg and Mac Lane [24] introduced categories and functors at all.

Definition 2.12 (Natural Transformation) *Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A* ***natural transformation*** *α from F to G is a family $\alpha = \{\alpha_C : F\,C \to G\,C\}_{C \in \mathcal{C}}$ of $\mathcal{D}$-morphisms such that, for all $A, B \in \mathcal{C}$ and $f : A \to B$, the following square commutes:*

$$\begin{array}{ccc} F\,A & \xrightarrow{\alpha_A} & G\,A \\ {\scriptstyle F\,f}\downarrow & & \downarrow{\scriptstyle G\,f} \\ F\,B & \xrightarrow[\alpha_B]{} & G\,B \end{array}$$

To indicate that α is a natural transformation from F to G, we write $\alpha : F \Rightarrow G$.

For example, let $\mathcal{C}$ be a category, $A, B \in \mathcal{C}$ and $f : A \to B$. Consider the hom-functors $\mathcal{C}(-, A)$ and $\mathcal{C}(-, B)$. Define a natural transformation $\alpha : \mathcal{C}(-, A) \Rightarrow \mathcal{C}(-, B)$, where $\alpha_C := f \circ -$, i.e. α_C is postcomposition by f. It is easy to check that this defines a natural transformation. By abuse of notation, we will also write α simply as $f \circ -$ and call it postcomposition by f.

Natural transformations where each component is iso are called **natural isomorphisms** or **natural isos**.

If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, $F, G : \mathcal{D} \to \mathcal{E}$ and $H : \mathcal{C} \to \mathcal{D}$ functors, then a natural transformation $\alpha : F \Rightarrow G$ canonically induces a natural transformation $\alpha * H : F \circ H \Rightarrow G \circ H$ by setting $(\alpha * H)_A := \alpha_{H\,A}$. This is called **whiskering**.

Natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ can be composed, the composite $\beta \circ \alpha : F \Rightarrow H$ is defined pointwise. A routine calculation shows $(\beta \circ \alpha) * I = (\beta * I) \circ (\alpha * I)$ for any functor I of appropriate type. For two fixed categories $\mathcal{C}$ and $\mathcal{D}$, one gets a category with functors from $\mathcal{C}$ to $\mathcal{D}$ as objects and natural transformations as arrows:

Definition 2.13 (Functor Category) *Let $\mathcal{C}, \mathcal{D}$ be categories. The* ***functor category*** *$[\mathcal{C}, \mathcal{D}]$ is defined as follows:*

- *Objects of $[\mathcal{C}, \mathcal{D}]$ are functors $\mathcal{C} \to \mathcal{D}$.*
- *A morphism from F to G is a natural transformation $\alpha : F \Rightarrow G$.*
  - *The composition of $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ is defined pointwise.*
  - *The identity on F is given by $\{\mathrm{id}_{F\,A}\}_{A \in \mathcal{C}}$, which is trivially a natural transformation.*

For a category $\mathcal{C}$, elements of $[\mathcal{C}^{op}, \mathsf{Set}]$ are called **presheaves**. A very important functor is the the **Yoneda embedding**, used for the fullification (Section 4.5).

Definition 2.14 (Yoneda Embedding) *Let $\mathcal{C}$ be a category. The* ***Yoneda embedding*** *$\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{op}, \mathsf{Set}]$ is the functor defined as follows:*

- *On objects, define $\mathcal{Y}\, C := \mathcal{C}(-, C)$.*
- *On arrows, $f : C \to D$, $\mathcal{Y}\, f$ is postcomposition by f.*

The key result of elementary category theory is the **Yoneda lemma**, notably the only non-definition presented in this section. The following is the contravariant version of the Yoneda lemma, where, for functors $F, G : \mathcal{C} \to \mathcal{D}$, $\mathsf{Nat}(F, G)$ denotes the collection of all natural transformations $F \Rightarrow G$.

Theorem 2.15 (Yoneda Lemma) *Let $\mathcal{C}$ be a category and $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a functor. For each $A \in \mathcal{C}$, there exists a bijection*

$$\eta_{A,F} : \mathsf{Nat}(\mathcal{C}(-, A), F) \to FA.$$

In particular, the natural transformations $\alpha : \mathcal{C}(-, A) \Rightarrow F$ form a set. Further, these bijections are natural in A, and if $\mathcal{C}$ is small, these bijections are also natural in F.

A key consequence of the Yoneda lemma when applied to the Yoneda embedding $\mathcal{Y}$ is that $\mathcal{Y}$ is fully faithful, which is a crucial insight for the proof that the fullification is indeed a full functor (see Theorem 4.16).

Lemma 2.16 *$\mathcal{Y}$ is fully faithful.*

There is also a way to represent natural transformations diagrammatically, via **pasting diagrams**. A situation where $\mathcal{C}$ and $\mathcal{D}$ are categories with functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : F \Rightarrow G$ is denoted as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

This idea scales well to larger diagrams, see another example in Proposition 4.2. Pasting diagrams can also be used to express equalities between natural transformations. Consider a setting with categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, functors $F : \mathcal{C} \to \mathcal{D}$, $G, H : \mathcal{D} \to \mathcal{E}$, and $I, J : \mathcal{C} \to \mathcal{E}$ as well as natural transformations $\alpha : G \Rightarrow H$ and $\beta : I \Rightarrow J$. The equality $\beta = \alpha * F$ can be written as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "I"{name=U}] \arrow[r, bend right, "J"'{name=D}] & \mathcal{E} \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathcal{E} \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

A real-world example is present in Proposition 4.3. If a natural transformation in a pasting diagram is just the identity transformation, the transformation arrow ($\Rightarrow$) can be removed from the pasting diagram. Pasting diagrams where all natural transformations are identity transformations are in fact commutative diagrams. For a more detailed account of pasting diagrams, see for instance the contemporary textbook account by Johnson and Yau [36] on 2-categories [49, 4, 17].

Adjunctions. The last fundamental categorical concept used in this thesis are **adjunctions**. None of our proofs touch this concept, but it is needed to define symmetric monoidal closed categories (SMCCs). There, we will see that adjunctions be used to categorically formalise the universal property of the tensor product of vector spaces, see the first paragraph of Section 2.2.

Definition 2.17 (Right Adjoint) *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ as well as $G : \mathcal{D} \to \mathcal{C}$ be functors. F is said to be **right adjoint** to G if for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$, there are bijections*

$$\theta_{A,B} : \mathcal{C}(G\,B, A) \to \mathcal{D}(B, F\,A)$$

that are natural in both A and B.

There is also a notion of left adjoints which is obtained by swapping the roles of F and G in the definition above.

2.2 Monoidal Categories via String Diagrams

Recall the category $\mathsf{Vect}_{\mathbb{R}}$ of real vector spaces. The tensor product of real vector spaces is a functor $\otimes : \mathsf{Vect}_{\mathbb{R}} \times \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{R}}$ which has certain properties such as symmetry and associativity (up to isomorphism). Also, $V \otimes \mathbb{R} \cong V$. Further, the universal property of the tensor product implies $\mathsf{Lin}(V \otimes W, U) \cong \mathsf{Lin}(V, \mathsf{Lin}(W, U))$ in a natural way, where $\mathsf{Lin}(V, W)$ denotes the vector space of linear maps from V to W, i.e. the hom-set of V and W in $\mathsf{Vect}_{\mathbb{R}}$. In other words, for fixed W, the functor $U \mapsto \mathsf{Lin}(W, U)$ is right adjoint to the functor $V \mapsto V \otimes W$.

All these properties can be formulated in terms of category theory and lead to to notion of a **monoidal category** [3, 47]. For the definition of monoidal categories, we follow Perrone [55, Chapter 6].

Definition 2.18 (Monoidal Category) *A **monoidal category** is a tuple $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ consisting of the following data:*

- *a category $\mathcal{C}$,*
- *a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,*
- *an object $I \in \mathcal{C}$,*
- *natural isomorphisms $\lambda_A : I \otimes A \to A$, $\rho_A : A \otimes I \to A$, and $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$*

such that the following diagrams commute:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[dd, "\alpha_{A,I,B}"'] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"] & \\
& A \otimes B \\
A \otimes (I \otimes B) \arrow[ur, "\mathrm{id}_A \otimes \lambda_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dd, "\alpha_{A,B,C \otimes D}"] \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B,C,D}"] \arrow[dd, "\alpha_{A,B,C} \otimes \mathrm{id}_D"'] & \\
& A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[dr, "\alpha_{A,B \otimes C,D}"'] & \\
& A \otimes ((B \otimes C) \otimes D) \arrow[uu, "\mathrm{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

The isomorphisms λ, ρ, and α are called **monoidal structure isomorphisms**. In the following, we will write a monoidal category as $(\mathcal{C}, \otimes, \mathrm{I})$ and leave the structure isos implicit. If the structure isos are all identities, then $(\mathcal{C}, \otimes, \mathrm{I})$ is called a **strict monoidal category**.

A particular class of monoidal categories are **Cartesian categories**, i.e. categories with finite products. For example, Set becomes a monoidal category with the usual Cartesian product and $\{*\}$ as monoidal unit. This monoidal structure is *not* strict. Monoidal categories generalise Cartesian categories by not requiring the universal property of products, i.e. by not having projections.

There is also a notion of monoidal categories where $\otimes$ is symmetric.

Definition 2.19 (Symmetric Monoidal Category) *Let $(\mathcal{C}, \otimes, \mathrm{I})$ be a monoidal category. We say that $(\mathcal{C}, \otimes, \mathrm{I})$ is a **symmetric monoidal category** if there is a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ such that the following diagrams commute:*

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[dr, "\sigma_{A,B\otimes C}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[d, "\sigma_{A,B} \otimes \mathrm{id}_C"'] & & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[dr, "\alpha_{B,A,C}"'] & & B \otimes (C \otimes A) \\
& B \otimes (A \otimes C) \arrow[ur, "\mathrm{id}_B \otimes \sigma_{A,C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \otimes A \arrow[d, "\sigma_{B,A}"] \\
A \otimes B \arrow[ur, "\sigma_{A,B}"] \arrow[r, "\mathrm{id}"'] & A \otimes B
\end{tikzcd}$$

*The structure isomorphism σ is called **braiding**.*

The commutative diagrams for the structure isos are known as **coherence conditions** and they assert that any two ways to compose structure isos to get from one object to another yield the same isomorphism.

The fact that $\mathsf{Lin}(V \otimes W, U) \cong \mathsf{Lin}(V, \mathsf{Lin}(W, U))$ for vector spaces motivates the notion of **symmetric monoidal closed categories**.

Definition 2.20 (Symmetric Monoidal Closed Category, SMCC) *Let $(\mathcal{C}, \otimes, \mathrm{I})$ be a symmetric monoidal category. We say that $(\mathcal{C}, \otimes, \mathrm{I})$ is a **symmetric monoidal closed category (SMCC)** if for all $A \in \mathcal{C}$, the functor $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a right adjoint.*

This right adjoint will be denoted $A \multimap - : \mathcal{C} \to \mathcal{C}$.

There is then an isomorphism $\mathcal{C}(A \otimes B, C) \to \mathcal{C}(A, B \multimap C)$, natural in A, B, and C (naturality in B requires a separate proof and is not immediate from the definition, see e.g. Perrone [55, Proposition 6.5.16]). The object $B \multimap C$ is called **internal hom** from B to C as it internalises the morphisms from B to C into the category $\mathcal{C}$. Think of vector spaces and tensor products again: One can show that the internal hom is given by $\mathsf{Lin}(W, U)$, which is the vector space of morphisms of type $W \to U$, i.e. the linear maps from W to U can be seen as an object of the category of vector spaces.

To later introduce our main subject of study, the category of Banach categories, it will be useful to consider the category of all SMCCs with functors between them that strictly preserve the SMCC structure.

Definition 2.21 ($\mathsf{SMCCCat}$) *The category* $\mathsf{SMCCCat}$ *of all SMCCs is defined as follows:*

- *The objects of* $\mathsf{SMCCCat}$ *are SMCCs.*
- *A morphism* $F : (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \multimap_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, \multimap_{\mathcal{D}})$ *is a functor such that*

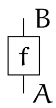
$$\begin{aligned} I_{\mathcal{D}} &= F\, I_{\mathcal{C}} \\ F\,A \otimes_{\mathcal{D}} F\,B &= F\,(A \otimes_{\mathcal{C}} B) \\ F\,A \multimap_{\mathcal{D}} F\,B &= F\,(A \multimap_{\mathcal{C}} B) \end{aligned}$$

  for all $A, B \in \mathcal{C}$ *and* F *preserves structure isos, i.e.* $F\lambda_A = \lambda_{F\,A}$ *and simularly for* $\rho, \alpha,$ *and* σ.

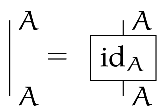
  – *Composition and identities are given by composition of functors and the identity functors, respectively.*

To reason about arrows in monoidal categories, the graphical calculus of string diagrams [53] is useful and used a lot in this thesis. In the following, key features are presented, following the development of Melliès [50].

Fix a symmetric monoidal category $(\mathcal{C}, \otimes, I)$. A morphism $f : A \to B$ in $\mathcal{C}$ is depicted as follows:

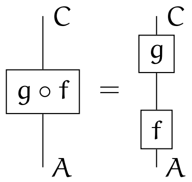


Sometimes, in larger diagrams, we draw diagrams from left to right instead of bottom-up. The identity is depicted without a box, i.e. the following diagrams are equal:

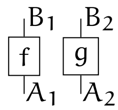


Composition is depicted by drawing boxes in sequence, i.e. if $f : A \to B$ and $g : B \to$

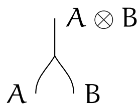
C, then



Monoidal products are represented by drawing wires in parallel, e.g. if $f : A_1 \to B_1$ and $g : A_2 \to B_2$, then $f \otimes g : A_1 \otimes A_2 \to B_1 \otimes B_2$ is drawn as follows:

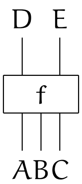


Wires can be merged as follows:

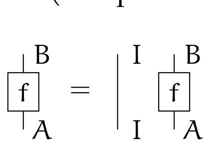


The diagram above represents the identity on $A \otimes B$ but translates between the two ways to draw it. This is sometimes convenient within larger diagrams. A similar gadget to split instead of merge wires can be used as well.

A general morphism between tensors, e.g. $f : A \otimes B \otimes C \to D \otimes E$ is drawn like this:

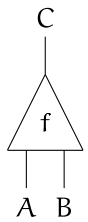


Note that in the diagram above, $(A \otimes B) \otimes C$ is identified with $A \otimes (B \otimes C)$, i.e. α is implicitly baked into string diagrams. In general, we will drop parentheses in terms such as $(A \otimes B) \otimes C$, appealing to the fact that all ways to parenthesise this expression are isomorphic in a coherent way. Also the unitors λ and ρ are built into the calculus (see next diagram). As a result, string diagrams represent morphisms *modulo composition with structure isos* (except for the braiding, see below).



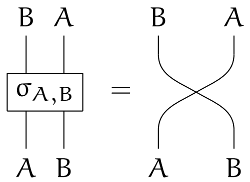
For the special case where the codomain of f only consists of one object, e.g. f :

$A \otimes B \to C$, we also use the following triangle notation:



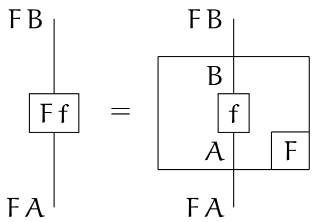
There is, however, nothing wrong with drawing f as a box.

The braiding σ is represented by crossing the wires:

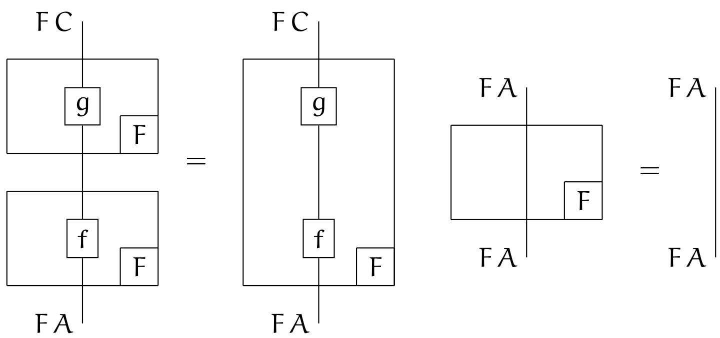


Labels on wires representing the domain and codomain categories may be dropped if they can be inferred from the context.

One can also represent functors graphically; this calculus is due to Cocket and Seely [7] and extends string diagrams. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ be symmetric monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. The arrow $F f : F A \to F B$ is drawn within a box labelled F as follows:



Everything outside the box is in the category $\mathcal{D}$, and everything inside the box is in the category $\mathcal{C}$. The functorial equations $F g \circ F f = F(g \circ f)$ and $F\, \mathrm{id}_A = \mathrm{id}_{F A}$ can then be written as follows:



Of particular interest in this thesis are **lax monoidal functors**.

Definition 2.22 (Lax Monoidal Functor) *Let* $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ *and* $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ *be monoidal categories. A functor* $F : \mathcal{C} \to \mathcal{D}$ *is **lax monoidal** if there are* $\mathcal{D}$*-morphisms* $\varepsilon : I_{\mathcal{D}} \to F\, I_{\mathcal{C}}$ *and* $\mu_{A,B} : F\,A \otimes_{\mathcal{D}} F\,B \to F\,(A \otimes_{\mathcal{C}} B)$*, where* μ *is natural in* A, B*, such that the following diagrams commute:*

$$
\begin{tikzcd}
(F\,A \otimes_{\mathcal{D}} F\,B) \otimes_{\mathcal{D}} F\,C \arrow[r, "\alpha_{F\,A,F\,B,F\,C}"] \arrow[d, "\mu_{A,B} \otimes_{\mathcal{D}} \mathrm{id}_{F\,C}"'] & F\,A \otimes_{\mathcal{D}} (F\,B \otimes_{\mathcal{D}} F\,C) \arrow[d, "\mathrm{id}_{F\,A} \otimes_{\mathcal{D}} \mu_{B,C}"] \\
F\,(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{D}} F\,C \arrow[d, "\mu_{A \otimes_{\mathcal{C}} B, C}"'] & F\,A \otimes_{\mathcal{D}} F\,(B \otimes_{\mathcal{C}} C) \arrow[d, "\mu_{A, B \otimes_{\mathcal{C}} C}"] \\
F\,((A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C) \arrow[r, "F\,\alpha_{A,B,C}"'] & F\,(A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C))
\end{tikzcd}
$$

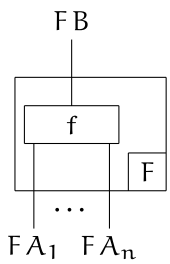
$$
\begin{tikzcd}
F\,A \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r, "\rho_{F\,A}"] \arrow[d, "\mathrm{id}_{F\,A} \otimes_{\mathcal{D}} \varepsilon"'] & F\,A \\
F\,A \otimes_{\mathcal{D}} F\,I_{\mathcal{C}} \arrow[r, "\mu_{A,I_{\mathcal{C}}}"'] & F\,(A \otimes_{\mathcal{C}} I_{\mathcal{C}}) \arrow[u, "F\,\rho_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{\mathcal{D}} \otimes_{\mathcal{D}} F\,A \arrow[r, "\lambda_{F\,A}"] \arrow[d, "\varepsilon \otimes_{\mathcal{D}} \mathrm{id}_{F\,A}"'] & F\,A \\
F\,I_{\mathcal{C}} \otimes_{\mathcal{D}} F\,A \arrow[r, "\mu_{I_{\mathcal{C}},A}"'] & F\,(I_{\mathcal{C}} \otimes_{\mathcal{C}} A) \arrow[u, "F\,\lambda_A"']
\end{tikzcd}
$$

*These diagrams are called **coherence diagrams**. The diagram on top is the **associativity diagram**, and the other two diagrams are called **left diagram** and **right diagram**, respectively. The morphisms* ε *and* $\mu_{A,B}$ *are called **coherence morphisms**.*

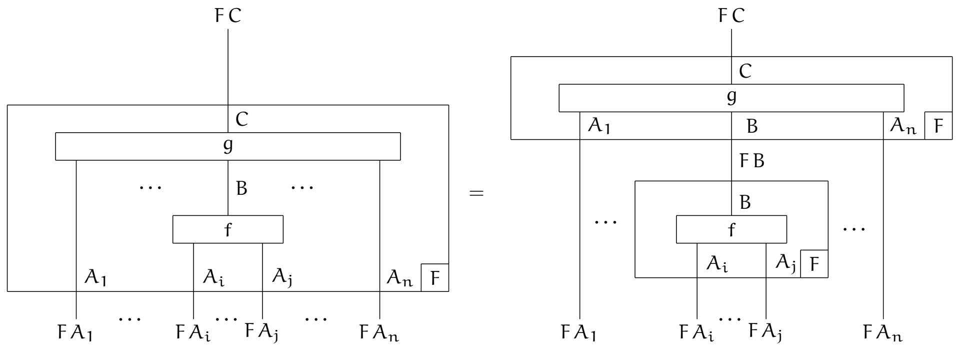
There is also a notion of a **colax monoidal** functor, where the coherence morphisms go in the other direction. A lax monoidal functor whose coherence morphisms are isos is called a **strong monoidal functor**. A strong monoidal functor is both lax and colax.

The coherence diagrams assert that any way to define a morphism $F\,A_1 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} F\,A_n \to F\,(A_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} A_n)$ using the coherence morphisms yield the same morphism, denoted $\mu_{A_1,\ldots,A_n}$.

If $f : A_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} A_n \to B$ is a $\mathcal{C}$-morphism, then $F\,f \circ \mu_{A_1,\ldots,A_n}$ is denoted as follows (where F is lax monoidal):



Coherence implies that the following diagrams are equal, see Melliès [50]. This is used in Lemma 4.14.



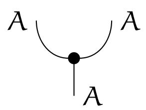
(2.1)

In general, we treat two diagrams as equal if they are equal up to structure isos and invertible coherence morphisms.

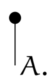
2.3 Markov Categories

Markov categories [5, 27] are a generalisation of categories of stochastic matrices as well as categories of Markov kernels, first axiomatised by Cho and Jacobs [5] and extensively studied by Fritz [27]. These two examples appear below (as FinStoch and Stoch). Markov categories are very useful in the categorical and synthetic approach to probability theory, where technical details of models (e.g. measurable spaces) are hidden and instead, crisp axioms are used to characterise the key properties of these models, allowing for more abstract and concise reasoning. Markov categories come with copy ($\mathsf{copy}_A : A \to A \otimes A$) and discard ($\mathsf{del}_A : A \to I$) morphisms, subject to certain natural conditions.

In diagrams, copy_A is written as



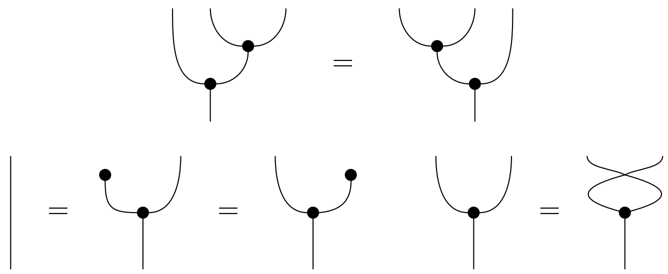
and del_A as



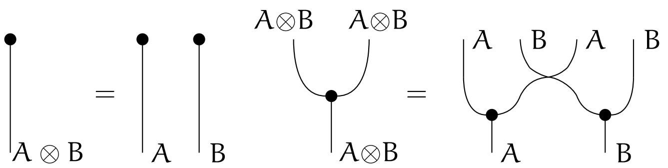
The definition of Markov categories is given diagrammatically.

Definition 2.23 (Markov Category, cf. Fritz [27, Definition 2.1]) *Let* $(\mathcal{C}, \otimes, I)$ *be a symmetric monoidal category.* $\mathcal{C}$ *is a* ***Markov category*** *if for all* $A \in \mathcal{C}$ *there are morphisms* $\mathsf{copy}_A : A \to A \otimes A$ *and* $\mathsf{del}_A : A \to I$ *such that*

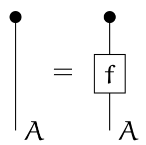
- copy_A *and* del_A *satisfy the* ***commutative comonoid equations****, i.e.*



- copy_A *and* del_A *are compatible with the monoidal structure, i.e.*



- del_A *is natural in* A*, i.e. for all* $f : A \to B$*, it holds that*



A particularly straightforward, but degenerate, example of a Markov category is a Cartesian category, i.e. a category with finite products: In such a category, the monoidal unit is a terminal object, and the delete morphism is then the unique arrow into this terminal object. In the notation of Definition 2.3, the copy operation $X \to X \times X$ is given by $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$. The axioms of a Markov category are easy to check.

The prime example of Markov categories, however, are Markov kernels (Stoch, see Definition 2.25). For simplicity, we first give the example of Markov kernels between finite sets ($\mathsf{FinStoch}$), which are exactly stochastic matrices. For both examples, the development of Fritz [27] is followed.

Definition 2.24 ($\mathsf{FinStoch}$) *The category* $\mathsf{FinStoch}$ *is defined as follows:*

- *The objects of* $\mathsf{FinStoch}$ *are finite sets.*
- *A morphism* $f : X \to Y$ *is a stochastic matrix* $f = (f_{x,y})_{x \in X, y \in Y}$*, i.e.* $f_{x,y} \in [0, 1]$ *for all* x, y *and* $\sum_{y \in Y} f_{x,y} = 1$ *for all* x*.*
  - *Composition is given by matrix multiplication.*

- *The identity on* X *is given by the identity matrix.*

To simplify notation, define $f(y|x) := f_{x,y}$ (sic!). The value $f(y|x)$ denotes the probability that the kernel f outputs y on input x.

FinStoch becomes a Markov category as follows:

- The monoidal product of sets X, Y is $X \otimes Y := X \times Y$.
- The monoidal product of matrices $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ is given by

$$(f \otimes g)((y, y')|(x, x')) = f(y|x) \cdot g(y'|x').$$

- The copy operation $\mathsf{copy}_X : X \to X \times X$ is given as

$$\mathsf{copy}_X((x_1, x_2)|x) = \begin{cases} 1 & \text{if } x = x_1 = x_2 \\ 0 & \text{otherwise.} \end{cases}$$

- The delete operation $\mathsf{del}_X : X \to \{0\}$ is given as

$$\mathsf{del}_X(0|x) := 1.$$

The category FinStoch can be generalised to the continuous case by using Markov kernels. The resulting category is called Stoch. The example below involves some elementary measure theory, see for instance the textbook by Cohn [8] for the corresponding definitions.

Definition 2.25 (Stoch) *The category* Stoch *is defined as follows:*

- *Objects of* Stoch *are measurable spaces, i.e. pairs* (X, Σ_X) *where* X *is a set and* Σ_X *a* σ*-algebra on* X*.*
- *A morphism* $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ *is a Markov kernel, i.e. a function* $f : \Sigma_Y \times X \to [0, 1]$ *such that for all* $x \in X$*,* $f(-, x) : \Sigma_Y \to [0, 1]$ *is a probability measure on* (Y, Σ_Y)*, and that for all* $A \in \Sigma_Y$*,* $f(A, -) : X \to [0, 1]$ *is Borel-measurable. The value* $f(A, x)$ *is also written as* $f(A|x)$*.*
  - *The composition of Markov kernels* $f : \Sigma_Y \times X \to [0, 1]$ *and* $g : \Sigma_Z \times Y \to [0, 1]$ *is defined as*

$$(g \circ f)(A|x) := \int_{y \in Y} g(A|y)\, f(dy|x).$$

This is a variant of the ***Chapman-Kolmogorov equation****.*

– *The identity on* (X, Σ_X) *is given by* $\mathrm{id}_X : \Sigma_X \times X \to [0, 1]$, *where*

$$\mathrm{id}_X(A|x) := \begin{cases} 1 & \textit{if } x \in A \\ 0 & \textit{otherwise.} \end{cases}$$

The integral in the Chapman-Kolmogorov equation is a Lebesgue integral, where $f(-|x) : \Sigma_Y \to [0, 1]$ is the probability measure used for the integration.

Stoch becomes a monoidal category as follows:

- The monoidal product of measurable spaces (X, Σ_X) and (Y, Σ_Y) is $(X \times Y, \Sigma_X \otimes \Sigma_Y)$, where $\Sigma_X \otimes \Sigma_Y$ is the product σ-algebra, which is generated by $\{A \times B \mid A \in \Sigma_X, B \in \Sigma_Y\}$. The monoidal unit is $(\{*\}, \{\emptyset, \{*\}\})$.
- The monoidal product of Markov kernels $f : \Sigma_{Y_1} \times X_1 \to [0, 1]$ and $g : \Sigma_{Y_2} \times X_2 \to [0, 1]$ is given by

$$(f \otimes g) : (\Sigma_{Y_1} \otimes \Sigma_{Y_2}) \times (X_1 \times X_2) \to [0, 1],$$

  where, on the generating sets of $\Sigma_{Y_1} \otimes \Sigma_{Y_2}$,

$$(f \otimes g)((A \times B)|(x_1, x_2)) = f(A|x_1) \cdot g(B|x_2).$$

- The copy operation $\mathsf{copy}_{(X,\Sigma_X)} : (\Sigma_X \otimes \Sigma_X) \times X \to [0, 1]$ is defined, on the generating sets of $\Sigma_X \otimes \Sigma_X$, as

$$\mathsf{copy}_{(X,\Sigma_X)}(A \times B|x) = \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{otherwise.} \end{cases}$$

- The delete operation $\mathsf{del}_{(X,\Sigma_X)} : \{\emptyset, \{*\}\} \times X \to [0, 1]$ is given by

$$\mathsf{del}_{(X,\Sigma_X)}(A|x) := \begin{cases} 0 & \text{if } X = \emptyset \\ 1 & \text{if } X = \{*\}. \end{cases}$$

Note that in the definition above, the equations for $f \otimes g$ and $\mathsf{copy}_{(X,\Sigma_X)}$ do not serve as defining equations as they only say how the respective probability measures behave on rectangles of measurable sets. But one can show via a standard measure theoretic argument (see e.g. Cohn [8, Corollary 1.6.3]) that any two probability measures that agree on these rectangles already agree on the entire product σ-algebra. A formal way to define these functions is via the Giry monad [29]. See Fritz' development for details.

The category FinStoch is a full subcategory of Stoch: The subcategory embedding sends a finite set X to $(X, \mathcal{P}(X))$, where $\mathcal{P}(X)$ is the powerset of X. A stochastic matrix f can be seen as a Markov kernel by interpreting $f(\{y_1, \dots, y_k\}|x)$ as $\sum_{i=1}^{k} f(y_i|x)$.

A natural way to obtain Markov categories is from Kleisli categories [40] of monoidal monads (see Fritz [27, Section 3]). The Giry monad [29] mentioned above is a particular example of a monoidal monad, and its Kleisli category gives rise to Stoch.

As for SMCCs, it will be useful to consider the category of all Markov categories.

Definition 2.26 (MarkovCat) *The category* MarkovCat *of all Markov categories is defined as follows:*

- *The objects of* MarkovCat *are Markov categories.*
- *A morphism* $F : (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \mathsf{copy}, \mathsf{del}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, \mathsf{copy}', \mathsf{del}')$ *is a functor such that*

$$\begin{aligned} I_{\mathcal{D}} &= F\, I_{\mathcal{C}} \\ F\, A \otimes_{\mathcal{D}} F\, B &= F\,(A \otimes_{\mathcal{C}} B) \\ F\, \mathsf{copy}_A &= \mathsf{copy}'_{F\,A} \\ F\, \mathsf{del}_A &= \mathsf{del}'_{F\,A} \end{aligned}$$

  for all $A, B \in \mathcal{C}$ *and* F *preserves structure isos, i.e.* $F\,\lambda_A = \lambda_{F\,A}$ *and similarly for* ρ, α, *and* σ.

  - *Composition and identities are given by composition of functors and the identity functors, respectively.*

The full subcategory of small Markov categories is denoted $\mathsf{MarkovCat}_{\mathsf{small}}$.

2.4 Banach Categories

After having introduced the necessary preliminaries, we are now in the position to introduce our main subject of study: **Banach categories**. The terminology is novel. A Banach category is a triple consisting of a Markov category, an SMCC, and a lax monoidal functor from the Markov category to the SMCC. For example, FinStoch and finite dimensional vector spaces form a Banach category, as noted in Section 3.3.1. Azevedo de Amorim [15] uses these triples to give semantics to his PPL unifying linear languages and Markov kernel languages, observing that there are many models of interest. For instance, he observes that Banach categories often arise from Kleisli categories [40] and Eilenberg-Moore algebras [25] of monoidal monads, with a canonical lax monoidal functor between them. As pointed out earlier, a prominent such monad is the Giry monad [29] whose Kleisli category is just

Stoch. Motivated by the fact that there are many natural Banach categories, we proceed to give a general treatment of them. The name is justified by the fact that the SMCC of a Banach category is often some kind of closed normed vector space, i.e. a Banach space (see e.g. the examples in Sections 3.3.1 and 3.3.2, particularly the latter).

Definition 2.27 (Banach Category) *A **Banach category** is a triple $(\mathcal{C}, \mathsf{F}, \mathcal{L})$ where $\mathcal{C}$ is a Markov category, $\mathcal{L}$ an SMCC, and $\mathsf{F} : \mathcal{C} \to \mathcal{L}$ a lax monoidal functor that is faithful.*

The requirement that F is faithful comes from the fact that F is faithful in the interesting examples, as noted by Azevedo de Amorim [15]. Banach categories do form a category, which we will denote BanCat.

Definition 2.28 (Banach Category) *The category* BanCat *of all Banach categories is defined as follows:*

- *The objects of* BanCat *are Banach categories.*
- *A morphism* $\mathsf{f} : (\mathcal{C}, \mathsf{F}, \mathcal{L}) \to (\mathcal{C}', \mathsf{F}', \mathcal{L}')$ *is a pair* $\mathsf{f} = (\mathsf{G}, \mathsf{H})$ *where* G *is an arrow in* MarkovCat *and* H *is an arrow in* SMCCCat *such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathsf{F}"] \arrow[d, "\mathsf{G}"'] & \mathcal{L} \arrow[d, "\mathsf{H}"] \\ \mathcal{C}' \arrow[r, "\mathsf{F}'"'] & \mathcal{L}' \end{tikzcd}$$

  - *Composition is defined componentwise, i.e.* $(\mathsf{G}', \mathsf{H}') \circ (\mathsf{G}, \mathsf{H}) := (\mathsf{G}' \circ \mathsf{G}, \mathsf{H}' \circ \mathsf{H})$.
  - *The identity on* $(\mathcal{C}, \mathsf{F}, \mathcal{L})$ *is given by* $\mathrm{id}_{(\mathcal{C}, \mathsf{F}, \mathcal{L})} := (\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{L}})$.

The full subcategory of BanCat *consisting of Banach categories where* F *is full is denoted* $\mathsf{BanCat}_{\mathsf{full}}$.

Considering the category of all Banach categories is justified by the fact the the fulification of a Banach category, our main theorem, gives rise to a functor on the category of Banach categories (Proposition 4.30), which opens up potential future work to study the properties of this functor, see Section 5.3. Another example morphism in BanCat is from the Banach category consisting of FinStoch and finite dimensional vector spaces to the Banach category of Stoch and regularly ordered Banach spaces (see Sections 3.3.1 and 3.3.2 for the examples): Both functors are just the inclusion functors which preserve the respective structures strictly.

Chapter 3

A Graphical Language for Banach Categories

In this chapter, we give an overview of Banach categories in view of string diagrams.

First, we motivate this approach by studying the example of convex spaces where string diagrams allow for an easy to parse presentation of the definitions involved (Section 3.1). Then, we study lax monoidal functors diagrammatically and also give a brief outlook to colax monoidal functors (Section 3.2). This chapter is rounded up by the discussion of two examples: $\mathsf{FinStoch}$ and finite dimensional vector spaces as well as Stoch and regularly ordered Banach spaces (Section 3.3).

3.1 Motivation: Diagrammatic Description of Convex Spaces

We now investigate convex spaces in light of string diagrams. Convex spaces (Definition 3.1) can be seen as a generalisation of real or convex vector spaces. The motivation for this investigation is two-fold: Firstly, this small example nicely highlights how string diagrams can be used keep an overview of abstract definitions, and secondly, the SMCC of a Banach category is often some kind of category of Banach spaces, which admit a convex structure. In the future, one could add convexity conditions to the definition of a Banach category to see whether this yields interesting properties (see Section 5.3).

Recall the notion of convex subsets of $\mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is called **convex** if for all $x, y \in U$ and $\lambda \in [0, 1]$, it holds that $\lambda x + (1-\lambda)y \in U$. Geometrically, this means that the straight line connecting x and y is fully contained in U.

The notion of a **convex space** is based on this concept, but far more general. Stone was the first to axiomatically characterise convex subsets of vector spaces [59], which ultimately led to the modern definition of convex spaces. We follow the notation by Fritz [26], who also outlines the complicated history of convex spaces.

A convex space is a set $\mathcal{S}$ together with a family of functions $\{\mathsf{cc}_\lambda : \mathcal{S} \times \mathcal{S} \to \mathcal{S}\}_{\lambda \in [0,1]}$ such that certain natural properties hold. Here, the functions cc_λ model **convex combinations**. For instance, if $U \subseteq \mathbb{R}^n$ is a convex set, then $\mathsf{cc}_\lambda(v, w) := \lambda v + (1-\lambda)w$

gives rise to a convex space.

Definition 3.1 (Convex Space, cf. Fritz [26, Definition 3.1]) *Let $\mathcal{S}$ be a set and let $\{cc_\lambda : \mathcal{S} \times \mathcal{S} \to \mathcal{S}\}_{\lambda \in [0,1]}$ be a family of functions. The pair $(\mathcal{S}, \{cc_\lambda\}_\lambda)$ is a* ***convex space*** *if the following properties hold:*

- *Unit law:*
$$cc_0(x, y) = x,$$
- *Idempotency:*
$$cc_\lambda(x, x) = x$$
- *Parametric commutativity:*
$$cc_\lambda(x, y) = cc_{1-\lambda}(y, x)$$
- *Deformed parametric associativity:*
$$cc_\lambda(cc_\mu(x, y), z) = cc_{\tilde{\lambda}}(x, cc_{\tilde{\mu}}(y, z)),$$
where $\tilde{\lambda} = \lambda\mu$ and
$$\tilde{\mu} = \begin{cases} \frac{\lambda(1-\mu)}{1-\lambda\mu} & \text{if } \lambda\mu \neq 1 \\ 0 & \text{otherwise, i.e. if } \lambda = \mu = 1 \end{cases}$$

All these axioms are natural, except for deformed parametric associativity. However, it is easily seen to be a key property of the example of convex sets mentioned above, which motivates this axiom. Further, the choice that $\tilde{\mu} = 0$ if $\lambda = \mu = 1$ is arbitrary, one could set $\tilde{\mu}$ to any number from $[0, 1]$.

Convex sets are not the only examples of convex spaces. Even more, there are convex spaces that are not even subsets of a real or convex vector space.

Example 3.2 (Two-Point Space, cf. Fritz [26, Example 5.1]) *The pair $(\{1, 2\}, \{cc_\lambda\}_\lambda)$ is a convex space, where*
$$cc_\lambda(1, 2) := \begin{cases} 1 & \text{if } \lambda = 0 \\ 2 & \text{otherwise.} \end{cases}$$

In the example above, the values $cc_\lambda(2, 1)$ are determined by parametric commutativity.

Fritz distinguishes three types of convex spaces: **Geometric** ones which are subsets of a vector space, **combinatorial** ones in which, for all points x, y, the function

$cc_\lambda(x, y)$ is constant for $\lambda \in (0, 1)$, and mixed ones that combine both. Example 3.2 is the minimal nontrivial example for combinatorial convex spaces.

As one might expect, convex spaces form a category. Arrows in this category are **convex maps**, which will be defined below. To ease notation, we write $\lambda x + (1-\lambda)y$ for $cc_\lambda(x, y)$ in the following and call a set $\mathcal{S}$ a convex space if the convex combination operation can be inferred from the context.

Definition 3.3 (Convex Map, cf. Fritz [26, Definition 3.2]) *Let $\mathcal{S}$ and $\mathcal{S}'$ be two convex spaces. A function $f : \mathcal{S} \to \mathcal{S}'$ is a **convex map** if*

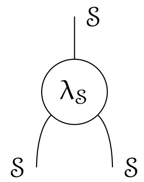
$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in \mathcal{S}$ and $\lambda \in [0, 1]$, i.e. f commutes with the convex combination operation.

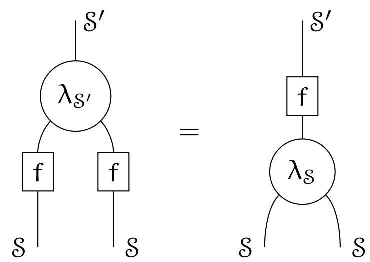
Definition 3.4 (ConvSpc, cf. Fritz [26, Definition 3.2]) *The category* ConvSpc *consists of convex spaces and convex maps.*

Checking that ConvSpc is indeed a category is routine. It turns out that the condition in Definition 3.3 can be represented diagrammatically in a way that makes the point of the equality immediately clear. We are not aware that the following diagrammatic descriptions appear in the existing literature.

For this, a diagrammatic version of the convex combination operation is needed. This is achieved by having a node labelled $\lambda_\mathcal{S}$ for all $\lambda \in [0, 1]$, where $\mathcal{S}$ is the underlying set of the convex space. The node labelled $\lambda_\mathcal{S}$ has two inputs and one output, as demonstrated in the following example:



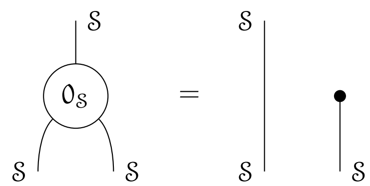
Two parallel wires in this diagram simply represent the cartesian product of sets. The property that $f : \mathcal{S} \to \mathcal{S}'$ is a convex map can then be represented diagrammatically as follows:



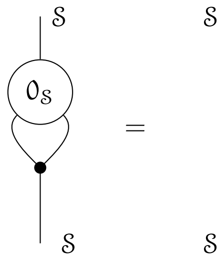
The above diagram immediately brings the point across: f commutes with the convex combination operation.

One can also think of formalising the conditions of Definition 3.1, but this requires copying, swapping, and delete morphisms. Since convex spaces are sets with Cartesian product structure, they clearly have copy and swap operations, which are given by the maps $x \mapsto (x, x)$ and $(x, y) \mapsto (y, x)$, respectively. The delete morphism $\mathcal{S} \to \{*\}$ is given by $x \mapsto *$. In fact, $\mathsf{ConvSpc}$ is even a Markov category since Set is.

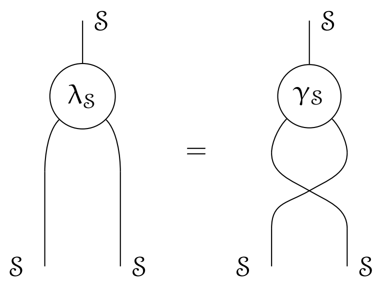
The unit law can be written as the following equality:



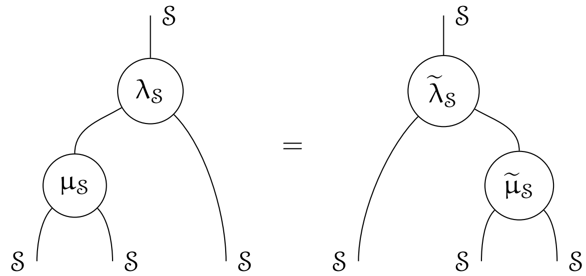
Here, the wire on the right represents the delete operation. The other wire on the right of the equals sign denotes the identity. The law of idempotency requires copying and can be represented as follows:



In the diagram above, the node where one wire enters and two wires exit denotes the copy operation. The law of parametric commutativity can be written as follows, where $\gamma = 1 - \lambda$:



The swap operation is represented by the crossing of two wires. Finally, deformed parametric associativity can be phrased as follows, where $\widetilde{\lambda}, \widetilde{\mu}$ are as in Definition 3.1:

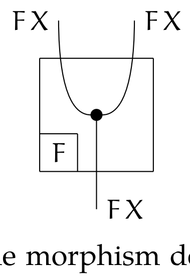


That is, convex spaces can be described entirely in terms of diagrams. In the future (see Section 5.3), one may investigate whether it is sensible to require that the SMCC of a Banach category has some kind of convex structure, motivated by the fact that these categories are often categories of Banach spaces (which are convex spaces).

3.2 Diagrammatic Description of Banach Categories

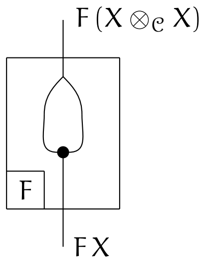
Lax monoidal functors (Definition 2.22) come with coherence morphisms relating the monoidal structures of their domain and codomain categories, and it is possible to characterise these maps diagrammatically. In the same spirit, and also motivated by the full diagrammatic description of convex spaces, one can ask whether the lax monoidal functor in a Banach category has additional properties that can be captured in terms of diagrams. In particular, one could investigate how the CD structure interacts with the coherence morphisms of the lax monoidal functor. For the remainder of this chapter, fix a Banach category consisting of a Markov category $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \mathsf{copy}, \mathsf{del})$, an SMCC $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, \multimap)$, and a lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ with coherence morphisms $\varepsilon : I_{\mathcal{D}} \to F\, I_{\mathcal{C}}$ and $\mu_{X,Y} : F\,X \otimes_{\mathcal{D}} F\,Y \to F\,(X \otimes_{\mathcal{C}} Y)$.

Interaction of CD structure and coherence morphisms. Unfortunately, it turns out that the CD structure and the coherence morphisms of F do not interact sufficiently well. For example, one might hope that, up to coherence or structure morphisms, there is a diagrammatic representation of $F\,\mathsf{copy}_X$ where one wire enters F's functor box and two wires exit, i.e. some diagram of the following shape:

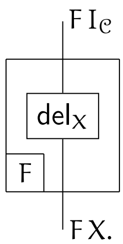


However, this is not possible: The morphism depicted in the above diagram has type $F\,X \to F\,X \otimes_{\mathcal{D}} F\,X$, while $F\,\mathsf{copy}_X$ has type $F\,X \to F\,(X \otimes_{\mathcal{C}} X)$. The coherence morphism $\mu_{Y,Z} : F\,Y \otimes_{\mathcal{D}} F\,Z \to F\,(Y \otimes_{\mathcal{C}} Z)$ cannot be postcomposed as it goes in the wrong direction. Also, there are not any other methods at hand to turn $F\,(X \otimes_{\mathcal{C}} X)$

into $\mathsf{F}\,\mathsf{X} \otimes_{\mathcal{D}} \mathsf{F}\,\mathsf{X}$. That is, the only diagrammatic way to appropriately represent $\mathsf{F}\,\mathsf{copy}_{\mathsf{X}}$ is as follows:

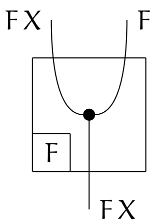


A similar issue occurs with $\mathsf{F}\,\mathsf{del}_{\mathsf{X}} : \mathsf{F}\,\mathsf{X} \to \mathsf{F}\,\mathrm{I}_{\mathcal{C}}$. There is no reasonable way to turn this into a morphism of type $\mathsf{F}\,\mathsf{X} \to \mathrm{I}_{\mathcal{D}}$, for a symmetric reason to the one outlined above. So the only diagram one gets is

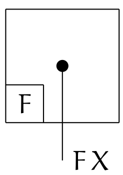


One can also investigate colax monoidal functors, where the coherence morphisms go in the other direction.

The case of colax monoidal functors. The issues in the previous paragraph can be avoided when F is colax monoidal, i.e. $\mu_{\mathsf{X},\mathsf{Y}}$ has type $\mathsf{F}\,(\mathsf{X} \otimes_{\mathcal{C}} \mathsf{Y}) \to \mathsf{F}\,\mathsf{X} \otimes_{\mathcal{D}} \mathsf{F}\,\mathsf{Y}$ and ε has type $\mathsf{F}\,\mathrm{I}_{\mathcal{C}} \to \mathrm{I}_{\mathcal{D}}$. This is because $\mu_{\mathsf{X},\mathsf{X}} \circ \mathsf{F}\,\mathsf{copy}_{\mathsf{X}}$ and $\varepsilon \circ \mathsf{del}_{\mathsf{X}}$ have types $\mathsf{F}\,\mathsf{X} \to \mathsf{F}\,\mathsf{X} \otimes_{\mathcal{D}} \mathsf{F}\,\mathsf{X}$ and $\mathsf{F}\,\mathsf{X} \to \mathrm{I}_{\mathcal{D}}$, respectively, which can be represented as follows:



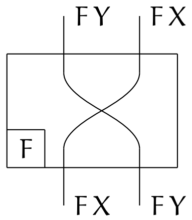
and



As usual, the coherence morphisms are omitted in the diagram and $\mathrm{I}_{\mathcal{D}}$ is represented without wire.

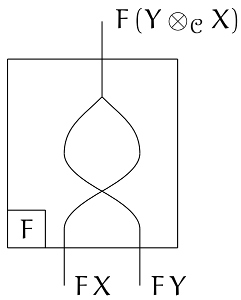
However, in most examples of interest for this thesis, the functor is lax and not colax. Further, there is no reason why the above observation should be useful, for instance by yielding a Markov structure on $\mathcal{D}$. Therefore, colax functors are not studied further in this thesis.

Interaction with the braiding. Another natural question to ask is whether there is any diagrammatic representation of $F\,\sigma_{X,Y}$, where σ is the braiding of $\mathcal{C}$. One might expect a diagram of the following shape (up to coherence morphisms):

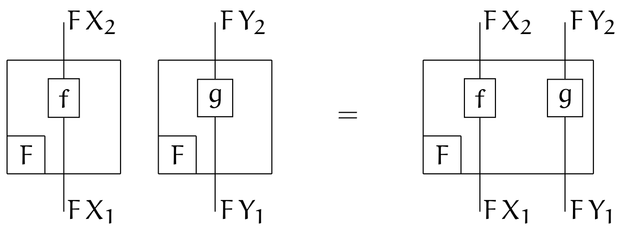


Observe that $F\,\sigma_{X,Y}$ has type $F\,(X \otimes_{\mathcal{C}} Y) \to F\,(Y \otimes_{\mathcal{C}} X)$. Using $\mu_{X,Y}$, this can be turned into a morphism $(F\,\sigma_{X,Y}) \circ \mu_{X,Y} : F\,X \otimes_{\mathcal{D}} F\,Y \to F\,(Y \otimes_{\mathcal{C}} X)$. However, since μ is not iso, this cannot be turned into a morphism of type $F\,X \otimes_{\mathcal{D}} F\,Y \to F\,Y \otimes_{\mathcal{D}} F\,X$, i.e. the diagram above is not valid in general. But if F is strongly monoidal, then μ is iso and $\mu^{-1}_{Y,X} \circ (F\,\sigma_{X,Y}) \circ \mu_{X,Y}$ has the desired type, i.e. the diagram above is valid.

In the general lax case, the only diagrammatic representation one gets is the one corresponding to $(F\,\sigma_{X,Y}) \circ \mu_{X,Y}$, which can be written as follows:



Parallel composition: Merging functor boxes. Let $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ be $\mathcal{C}$-morphisms. Are $F\,f \otimes_{\mathcal{D}} F\,g$ and $F\,(f \otimes_{\mathcal{C}} g)$ equal up to structure or invertible coherence morphisms, i.e. does the following equality of diagrams hold?



(3.1)

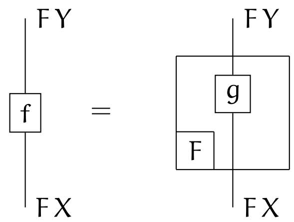
This is, in a slightly more general case, discussed by Melliès [50, Section 4]. The result is that the above equality holds when F is strongly monoidal. It follows from the following observation: Since $\mu_{A,B} : F\,A \otimes_{\mathcal{D}} F\,B \to F\,(A \otimes_{\mathcal{C}} B)$ is natural in both A and B, the following square commutes:

$$\begin{tikzcd}
F X_1 \otimes_{\mathcal{D}} F Y_1 \arrow[r, "\mu_{X_1,Y_1}"] \arrow[d, "F f \otimes_{\mathcal{D}} F g"] & F(X_1 \otimes_{\mathcal{C}} Y_1) \arrow[d, "F(f \otimes_{\mathcal{C}} g)"] \\
F X_2 \otimes_{\mathcal{D}} F Y_2 \arrow[r, "\mu_{X_2,Y_2}"] & F(X_2 \otimes_{\mathcal{C}} Y_2)
\end{tikzcd}$$

That is, $F(f \otimes_{\mathcal{C}} g) \circ \mu_{X_1,Y_1} = \mu_{X_2,Y_2} \circ (F f \otimes_{\mathcal{D}} F g)$. Since μ is a natural isomorphism, it holds that $F(f \otimes_{\mathcal{C}} g) = \mu_{X_2,Y_2} \circ (F f \otimes_{\mathcal{D}} F g) \circ \mu^{-1}_{X_1,Y_1}$. As string diagrams are operating modulo structure isos and invertible coherence morphisms, Equation (3.1) holds.

It is reasonable to require that $\varepsilon : I_{\mathcal{D}} \to F I_{\mathcal{C}}$ is iso since this is the case in many examples, see also Section 3.3. However, this is not the case for μ, see the example in Section 3.3.2.

Refinement of description if functor is full. If F is a full functor, then for all $X, Y \in \mathcal{C}$, the mapping $F : \mathcal{C}(X, Y) \to \mathcal{D}(F X, F Y)$ is surjective. In other words, for all $f : F X \to F Y$, there exists $g : X \to Y$ such that $F g = f$. This does not rely on any monoidality assumptions at all. In diagrams, this can be depicted as follows:



3.3 Examples of Banach Categories

Banach categories have important applications in categorical probability theory and beyond, as noted by Azevedo de Amorim [15]. In what follows, two examples are outlined, the first is a specialisation of the second to the finite case.

3.3.1 Stochastic Matrices and Finite Dimensional Vector Spaces

$\mathsf{FinStoch}$ is the Markov category of finite sets and stochastic matrices, see Definition 2.24. There is a strong monoidal functor to the category of finite dimensional vector spaces (denoted $\mathsf{FinDimVect}_{\mathbb{R}}$), which is an SMCC. Recall that an arrow $m : X \to Y$ in $\mathsf{FinStoch}$ is just a stochastic matrix with entries $m(y|x)_{x \in X, y \in Y}$.

Definition 3.5 *Define a functor* $F : \mathsf{FinStoch} \to \mathsf{FinDimVect}_{\mathbb{R}}$ *as follows:*

- *On objects,* $F X := \mathbb{R}^X$.
- *On arrows* $m : X \to Y$*, define* $F m : \mathbb{R}^X \to \mathbb{R}^Y$ *as*

$$(F m)(f)(y) := \sum_{x \in X} m(y|x) \cdot f(x).$$

It is routine to check that $F\,m : \mathbb{R}^X \to \mathbb{R}^Y$ is indeed a linear map and that F respects identities and composition. Therefore, F is a functor.

Recall that $\mathbb{R}$ is the monoidal unit of $\mathsf{FinDimVect}_{\mathbb{R}}$, while $\{*\}$ is the monoidal unit of $\mathsf{FinStoch}$. There is a trivial linear map $\varepsilon : \mathbb{R} \to F\{*\} = \mathbb{R}^{\{*\}}$, which is even iso. For the coherence morphism $\mu_{X,Y} : \mathbb{R}^X \otimes \mathbb{R}^Y \to \mathbb{R}^{X\times Y}$, it suffices to define a bilinear map $\gamma_{X,Y} : \mathbb{R}^X \times \mathbb{R}^Y \to \mathbb{R}^{X\times Y}$ thanks to the universal property of the tensor product.

The map $\gamma_{X,Y}$ is defined as follows:

$$\gamma_{X,Y}(f,g)(x,y) := f(x)\cdot g(y)$$

It is easily checked that $\gamma_{X,Y}$ is bilinear and natural in X, Y. A routine application of the universal property of the tensor product shows that also $\mu_{X,Y}$ is natural in X, Y.

We are now in the position to prove that F is strong monoidal.

Proposition 3.6 *The functor F from Definition 3.5 is strong monoidal.*

Proof For associativity, thanks to the universal property of $\otimes$, it suffices to show that the following diagram commutes, where $a((f,g),h) := (f,(g,h))$:

$$\begin{tikzcd}
(\mathbb{R}^X \times \mathbb{R}^Y) \times \mathbb{R}^Z \arrow[r, "a"] \arrow[d, "\gamma\times\mathrm{id}"] & \mathbb{R}^X \times (\mathbb{R}^Y \times \mathbb{R}^Z) \arrow[d, "\mathrm{id}\times\gamma"] \\
\mathbb{R}^{X\times Y} \times \mathbb{R}^Z \arrow[d, "\gamma"] & \mathbb{R}^X \times \mathbb{R}^{Y\times Z} \arrow[d, "\gamma"] \\
\mathbb{R}^{(X\times Y)\times Z} \arrow[r, "F\,\alpha_{X,Y,Z}"] & \mathbb{R}^{X\times(Y\times Z)}
\end{tikzcd}$$

Recall that $\alpha_{X,Y,Z}$ is an arrow in $\mathsf{FinStoch}$, i.e. a stochastic matrix. Its definition is

$$\alpha_{X,Y,Z}((x_1,(y_1,z_1)|((x_2,y_2),z_2))) = \begin{cases} 1, & \text{if } x_1 = x_2,\, y_1 = y_2, \text{ and } z_1 = z_2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, by definition,

$$\begin{aligned}
&(F\,\alpha_{X,Y,Z})(f)(x_1,(y_1,z_1)) \\
&= \sum_{((x_2,y_2),z_2)\in(X\times Y)\times Z} \alpha_{X,Y,Z}((x_1,(y_1,z_1))|((x_2,y_2),z_2)) \cdot f((x_2,y_2),z_2) \\
&= f((x_1,y_1),z_1)
\end{aligned}$$

From this observation it is routine to check that the diagram commutes.

For the left coherence diagram, it suffices to show that the following diagram commutes, where $s(r, f) := r \cdot f$:

$$\begin{tikzcd} \mathbb{R} \times \mathbb{R}^X \arrow[r, "\varepsilon \times \mathrm{id}"] \arrow[d, "s"] & \mathbb{R}^{\{*\}} \times \mathbb{R}^X \arrow[d, "\gamma"] \\ \mathbb{R}^X & \mathbb{R}^{\{*\} \times X} \arrow[l, "F\lambda_X"] \end{tikzcd}$$

If $(r, f) \in \mathbb{R} \times \mathbb{R}^X$, then $(\varepsilon \times \mathrm{id})(r, f) = (g, f)$, where $g(*) = r$. Further, $\gamma_{\{*\},X}(g, f)$ is the map given by $(*, x) \mapsto g(*) \cdot f(x) = r \cdot f(x)$. Now λ_X is an arrow $\{*\} \times X \to X$ in $\mathsf{FinStoch}$ such that

$$\lambda_X(x_1 | (*, x_2)) = \begin{cases} 1, & \text{if } x_1 = x_2 \\ 0, & \text{otherwise,} \end{cases}$$

so it is easy to check that $(F\lambda_X)(\gamma_{\{*\},X}(g, f))(x) = r \cdot f(x)$, i.e. the square commutes.

Verifying the right coherence diagram is similar.

For strong monoidality, note that ε is trivially iso. Further, for all X, Y, it holds that $\dim(\mathbb{R}^X \otimes \mathbb{R}^Y) = \dim(\ker(\mu_{X,Y})) + \dim(\mathrm{range}(\mu_{X,Y}))$ by standard linear algebra. Also, $\dim(\mathbb{R}^X \otimes \mathbb{R}^Y) = \dim(\mathbb{R}^X) \cdot \dim(\mathbb{R}^Y) = |X| \cdot |Y|$. Consider the functions $h_{x,y} : X \times Y \to \mathbb{R}$ for $(x, y) \in X \times Y$, where

$$h_{x,y}(x', y') = \begin{cases} 1 & \text{if } x = x' \text{ and } y = y' \\ 0 & \text{otherwise.} \end{cases}$$

By basic linear algebra, the collection $\{h_{x,y}\}_{x \in X, y \in Y}$ is a basis of $\mathbb{R}^{X \times Y}$. It is clear that each $h_{x,y}$ is in the image of $\gamma_{X,Y}$ and hence in the image of $\mu_{X,Y}$. Since $\mu_{X,Y}$ is a linear map, this means that $\mu_{X,Y}$ is surjective.

In other words, $\dim(\mathrm{range}(\mu_{X,Y})) = |X| \cdot |Y|$. Since also $|X| \cdot |Y| = \dim(\ker(\mu_{X,Y})) + \dim(\mathrm{range}(\mu_{X,Y}))$, this necessarily means that $\dim(\ker(\mu_{X,Y})) = 0$, i.e. $\mu_{X,Y}$ is injective and hence iso. $\square$

Note that F is faithful, but far from being full. Any linear map not represented by a stochastic matrix is not in its image, for instance the constant zero map.

Variations of this example. One can consider variations of this example where one restricts the set morphisms in $\mathsf{FinDimVect}_{\mathbb{R}}$ to a smaller class. If one restricts to just the positive maps preserving the unit ball, then F becomes full. Differently, if one restricts to the positive maps of 1-norm exactly 1, then F is not full, as witnessed by the linear map represented by $\begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$ with respect to the standard basis.

3.3.2 Markov Kernels and Regularly Ordered Banach Spaces

The example above can be generalised to continuous probability leading to a lax monoidal functor with domain Stoch. This functor is standard in the literature, see e.g. Azevedo de Amorim [15]. The codomain of this functor will again be a category of vector spaces, namely regularly ordered Banach spaces with regular linear functions (RoBan for short). To formally define RoBan (in Definition 3.11), a few preliminary definitions are needed. We follow the development of Dahlqvist and Kozen [10], the key definition is the one of a **regular ordered normed vector space** [13, 60].

Definition 3.7 (Ordered Vector Space) *An* ***ordered vector space*** *is a* $\mathbb{R}$*-vector space* $(V, +, 0)$ *with a partial order* $\leqslant$ *such that the following conditions are satisfied:*

- *For all* $u, v, w \in V$*, it holds that* $u \leqslant v \Rightarrow u + w \leqslant v + w$*.*
- *For all* $u, v \in V$ *and* $\lambda \geqslant 0$*, it holds that* $u \leqslant v \Rightarrow \lambda u \leqslant \lambda v$*.*

Definition 3.8 (Ordered Normed Vector Space) *An* ***ordered normed vector space*** *is an ordered vector space* $(V, +, 0, \leqslant)$ *with a norm* $\|\cdot\|$ *such that* $V^+ := \{v \mid v \geqslant 0\}$ *is closed with respect to the topology generated by* $\|\cdot\|$*.*

For example, $\mathbb{R}^n$ with the euclidean norm and the order $v \leqslant w$ iff $v_i \leqslant w_i$ for all i is an ordered normed vector space.

Definition 3.9 (Regular Ordered Normed Vector Space) *A* ***regular ordered normed vector space*** *is an ordered normed vector space* $(V, +, 0, \leqslant, \|\cdot\|)$ *such that, for all* $v, w \in V$*, both*

$$-w \leqslant v \leqslant w \Rightarrow \|v\| \leqslant \|w\|$$

and

$$\|v\| = \inf\{\|u\| \mid -u \leqslant v \leqslant u\}.$$

In this case, $\|\cdot\|$ *is called a* ***lattice norm****.*

A routine argument shows that also $\mathbb{R}^n$ with the ordering mentioned above is a regular ordered normed space.

Recall that a **Banach space** is a normed vector space which is complete with respect to its norm, which then gives a meaning to the term **regular ordered Banach space**. Since $\mathbb{R}^n$ with the euclidean norm is complete, $\mathbb{R}^n$ with the euclidean norm and the ordering mentioned above is a regular ordered Banach space. Even more, any finite dimensional vector space can be turned into a regular ordered Banach space.

There is also a notion of morphisms between regular ordered normed vector spaces. First, a **positive linear map** is a linear map $f : (V, +, 0, \leqslant) \to (W, +', 0', \leqslant')$ is such that $0 \leqslant v$ implies $0 \leqslant' f(v)$, used in the definition below:

Definition 3.10 (Regular Operator) *Let V and W be regular ordered normed vector space spaces. A linear map $f : V \to W$ is a* ***regular operator*** *if f is the difference $f = g - h$, where g, h are positive linear maps.*

For example, for a stochastic matrix $M \in \mathbb{R}^{n \times n}$, the linear map $v \mapsto Mv$ is positive, hence regular. Regular ordered Banach spaces and regular operators form a category, which is called RoBan.

Definition 3.11 (RoBan, cf. Dahlqvist and Kozen [10]) *The category RoBan is defined as follows:*

- *Objects of RoBan are regular ordered Banach spaces.*
- *A morphism in RoBan is a regular operator.*
  - *Composition is given by composition of functions.*
  - *Identities are identity functions.*

There is also a tensor product $\widehat{\otimes}$ on RoBan turning RoBan into an SMCC, its monoidal unit is $\mathbb{R}$ and its relevant universal property is that there exists a positive bilinear map[1] $\widehat{\otimes} : V \times U \to V \widehat{\otimes} U$ such that for all positive bilinear maps $f : V \times U \to W$, there exists a unique positive linear map $\widehat{f} : V \widehat{\otimes} U \to W$ such that:

$$\begin{tikzcd} V \times U \arrow[r, "\widehat{\otimes}"] \arrow[dr, "f"'] & V \widehat{\otimes} U \arrow[d, "\widehat{f}"] \\ & W \end{tikzcd}$$

A particular regular ordered Banach space is important to define the lax monoidal functor from Stoch to RoBan: Given a measurable space (X, Σ_X), a **finite signed measure** on this space is a map $\nu : \Sigma_X \to \mathbb{R}$ such that ν is σ-additive and $\nu(\emptyset) = 0$. The set of all signed measures on (X, Σ_X) becomes a vector space by pointwise addition, and can even be shown to be an object of RoBan.

Definition 3.12 (cf. Azevedo de Amorim [15, Theorem 12]) *Define a functor F of type $\mathsf{Stoch} \to \mathsf{RoBan}$ as follows:*

- *On objects, $F(X, \Sigma_X)$ is the regular ordered Banach space of finite signed measures on (X, Σ_X).*
- *On arrows, a Markov kernel $f : \Sigma_Y \times X \to [0, 1]$ is sent to the regular operator $Ff : F(X, \Sigma_X) \to F(Y, \Sigma_Y)$, where*

$$(Ff)(\nu)(A) := \int_{x \in X} f(A|x)\, \nu(dx).$$

[1] A positive bilinear map $f : V \times U \to W$ is a bilinear map such that $v \geqslant 0$ and $u \geqslant 0$ imply $f(v, u) \geqslant 0$.

Observe how nicely this definition extends the finite case from Definition 3.5 by using an integral instead of a sum.

Azevedo de Amorim verifies that this is indeed a well-defined lax monoidal functor. The definition of the coherence morphisms is sketched in the following. The morphism $\varepsilon : \mathbb{R} \to \mathsf{F}(\{*\}, \{\emptyset, \{*\}\})$ is trivial as a signed measure ν on $(\{*\}, \{\emptyset, \{*\}\})$ is uniquely determined by the value $\nu(\{*\}) \in \mathbb{R}$. To define the coherence morphism $\mu_{(X,\Sigma_X),(Y,\Sigma_Y)} : \mathsf{F}(X, \Sigma_X)\widehat{\otimes}\mathsf{F}(Y, \Sigma_Y) \to \mathsf{F}(X \times Y, \Sigma_X \otimes \Sigma_Y)$, one uses the universal property of $\widehat{\otimes}$, by which it suffices to define a positive bilinear map $\mathsf{F}(X, \Sigma_X) \times (Y, \Sigma_Y) \to \mathsf{F}(X \times Y, \Sigma_X \otimes \Sigma_Y)$, which is just given by the product measure, i.e. $(\nu, \upsilon) \mapsto \nu \otimes \upsilon$. To prove that the coherence diagrams commute, one uses the universal property of $\widehat{\otimes}$ by which it suffices to verify these diagrams on pure tensors (similar to how it is done for $\mathsf{FinStoch}$ in Proposition 3.6).

As opposed to the finite case with $\mathsf{FinStoch}$, the functor $\mathsf{F} : \mathsf{Stoch} \to \mathsf{RoBan}$ is not strong monoidal, as μ is not iso, as pointed out by Azevedo de Amorim [15] as well as Dahlqvist and Kozen [10] (but it is essentially an observation from functional analysis).

Two final remarks: F is not full, as for instance the zero maps are not in its image. Furthermore, one can show that there is an SMCC morphism from the finite dimensional vector spaces to RoBan. In particular, one can show that there is a morphism of Banach categories from the Banach category with $\mathsf{FinStoch}$ and finite dimensional vector spaces to the Banach category of Stoch and RoBan.

Chapter 4

Fullification of a Banach Category

In this chapter, we prove that a Banach category can be fullified in the following sense: For any Banach category $(\mathcal{C}, F, \mathcal{L})$ such that $\mathcal{C}$ is small, there exists Banach category $(\mathcal{C}, \widetilde{F}, \widetilde{\mathcal{L}})$ and a strict monoidal functor between SMCCs $q : \widetilde{\mathcal{L}} \to \mathcal{L}$ such that $\widetilde{F}$ is full such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & \widetilde{\mathcal{L}} \arrow[d, "q"] \\ & \mathcal{L} \end{tikzcd}$$

This result is the main contribution of this thesis. It is subject to Corollary 4.28.

We first give an overview over a related construction to fullify Cartesian functors between Cartesian categories (Section 4.1), which contains lots of ideas that were reused in the fullification proof presented in this thesis. We then give an overview of required background material needed to define the fullification: Comma categories (Section 4.2), multicategories (Section 4.3), as well as Day convolution and Kan extensions (Section 4.4). In Section 4.5, we define the fullification and prove that $\widetilde{F}$ is indeed full, and in Section 4.6 we show that the fullification can be equipped with the required monoidal structure.

To only understand the definition of the full functor, but not the argument why it is lax monoidal, Section 4.4 can be skipped. Also, Section 4.1 is not an essential read but recommended to understand where our fullification construction originates from.

4.1 Previous Work: Fullification of Cartesian Functor

In work on definability in the λ-calculus, a construction to fullify Cartesian functors arose in work by Alimohamed [1] and later Katsumata [39], inspired by previous work due to Jung and Tiuryn [37] (see also related work in Section 5.3). Roughly speaking, they prove the fullification result we are aiming for in the special case where the monoidal structure is given by actual products in the sense of Definition 2.3. The fullification construction presented in this theses borrows many ideas from Alimohamed and Katsumata, and it is therefore important to shed some light on Alimohamed and Katsumata's main technical points. We follow the comprehensive outline given by Katsumata [39].

Let L be a small category with finite products (called **Cartesian** category), $\mathcal{C}$ a small category with finite products, finite coproducts[1], and exponentials[2] (called **bi-Cartesian closed** category, short **bi-CCC** [44]), and $\mathsf{F} : \mathsf{L} \to \mathcal{C}$ a functor strictly preserving finite products (called **strict Cartesian** functor). Katsumata then defines a bi-CCC $\mathbb{K}_\mathsf{F}$, a full strict Cartesian functor $\mathsf{Def} : \mathsf{L} \to \mathbb{K}_\mathsf{F}$, as well as a bi-CC functor (i.e. a functor preserving finite (co)products and exponentials) $\mathsf{q} : \mathbb{K}_\mathsf{F} \to \mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{L} \arrow[r, "\mathsf{Def}"'] \arrow[dr, "\mathsf{F}"'] & \mathbb{K}_\mathsf{F} \arrow[d, "\mathsf{q}"] \\ & \mathcal{C} \end{tikzcd}$$

The category $\mathbb{K}_\mathsf{F}$ is obtained from a pullback construction (see Definition 2.5) as follows. Katsumata defines a poset Ctx_L as the finite lists of objects in L, i.e. $(\mathsf{Obj}(\mathsf{L}))^*$. The ordering is given by $\mathsf{L} \leqslant \mathsf{L}'$ iff L is a prefix of L', i.e. $\mathsf{L} = [A_1, \dots, A_n]$ and $\mathsf{L}' = [A_1, \dots, A_n, A_{n+1}, \dots, A_{n+m}]$ for some $m \geqslant 0$. Ctx_L is then treated as a category. An object of Ctx_L, i.e. a list of L-objects, can be seen as an L-object by simply taking the Cartesian product of its elements. This gives rise to a functor $|-| : \mathsf{Ctx}_\mathsf{L} \to \mathsf{L}^{op}$, where

$$|[A_1, \dots, A_n]| := A_1 \times \cdots \times A_n.$$

On the empty list, $|[]|$ is a terminal object of L. The unique arrow in Ctx_L from $[A_1, \dots, A_n]$ to $[A_1, \dots, A_n, A_{n+1}, \dots, A_{n+m}]$ is sent to a morphism of type

$$A_1 \times \cdots \times A_n \times A_{n+1} \times \cdots \times A_{n+m} \to A_1 \times \cdots \times A_n,$$

which just "strips off" the last m components using the projection morphisms.

$\mathbb{K}_\mathsf{F}$ is then the pullback of the following diagram:

$$\mathcal{C} \xrightarrow{\mathsf{H}_\mathsf{F}} [\mathsf{Ctx}_\mathsf{L}, \mathsf{Set}] \xleftarrow{p} \mathsf{Sub}([\mathsf{Ctx}_\mathsf{L}, \mathsf{Set}])$$

Here, $\mathsf{H}_\mathsf{F}\, C := \mathcal{C}(\mathsf{F}|-|, C)$. Further, $\mathsf{Sub}([\mathsf{Ctx}_\mathsf{L}, \mathsf{Set}])$ is the category of subobjects of presheaves on Ctx_L, and p is the corresponding subobject fibration. Subobjects are

[1]A coproduct is defined like a product, but with all arrows reversed.

[2]Exponentials can informally be thought of as internal homs in Cartesian categories.

a categorical generalisation of subsets, see e.g. Jacobs [35, Section 1.3]. See also Katsumata for more background on the involved categories and for further references.

The pullback diagram is then as follows:

$$\begin{tikzcd} \mathbb{K}_F \arrow[r, "r"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sub}([\mathsf{Ctx}_L, \mathsf{Set}]) \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "H_F"'] & [\mathsf{Ctx}_L, \mathsf{Set}] \end{tikzcd}$$

Using a technical construction, Katsumata defines a functor $\mathsf{Def} : L \to \mathbb{K}_F$. They also prove (in Proposition 2) that Def is full and satisfies $F = q \circ \mathsf{Def}$, i.e. the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\mathsf{Def}"'] \arrow[dr, "F"'] & \mathbb{K}_F \arrow[r, "r"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sub}([\mathsf{Ctx}_L, \mathsf{Set}]) \arrow[d, "p"] \\ & \mathcal{C} \arrow[r, "H_F"'] & [\mathsf{Ctx}_L, \mathsf{Set}] \end{tikzcd}$$

This way, Katsumata proved their desired fullification result.

In this thesis, we do not work with Cartestian categories, but with the more general case of monoidal categories. Still, the rough proof strategy is similar. However, instead of a pullback construction, we employ a glueing construction with a comma category [45], as described by Hyland and Schalk [33, Section 4]. Further, in addition to Alimohamed and Katsumata, we conjecture that our fullification construction has a universal property, see the first point of future work in Section 5.3.

4.2 Comma Categories

Comma categories are a standard construction in category theory, first developed by Lawvere in his 1963 PhD thesis (only published in 2004) [45, p. 36]. In the following, the development of Borceux is followed [2, Section 1.6], with slightly different notation.

Definition 4.1 (Comma Category, cf. Borceux [2, Definition 1.6.1]) *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories and $F : \mathcal{A} \to \mathcal{C}$, $G : \mathcal{B} \to \mathcal{C}$ functors. The* ***comma category*** *of F and G, written $F \downarrow G$, is defined as follows:*

- *Objects of $F \downarrow G$ are triples (A, f, B), where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $f \in \mathcal{C}(F\,A, G\,B)$.*
- *Morphisms of $F \downarrow G$ from (A, f, B) to (A', g, B') are pairs (a, b) such that $a \in \mathcal{A}(A, A')$, $b \in \mathcal{B}(B, B')$, and the following diagram commutes:*

$$\begin{tikzcd} F\,A \arrow[r, "f"] \arrow[d, "F\,a"'] & G\,B \arrow[d, "G\,b"] \\ F\,A' \arrow[r, "g"'] & G\,B' \end{tikzcd}$$

– *Composition and identities in* $F \downarrow G$ *are componentwise, i.e.* $(a', b') \circ (a, b) = (a' \circ a, b' \circ b)$ *and* $\mathrm{id}_{(A,f,B)} = (\mathrm{id}_A, \mathrm{id}_B)$.

The objects of the comma category $F \downarrow G$, where $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, are triples of two objects from $\mathcal{A}$ and $\mathcal{B}$, respectively, and a morphism $F\,A \to G\,B$. It turns out that there are canonical projection functors $F \downarrow G \to A$ and $F \downarrow G \to B$.

Proposition 4.2 (cf. Borceux [2, Proposition 1.6.2]) *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *be categories and* $F : \mathcal{A} \to \mathcal{C}$, $G : \mathcal{B} \to \mathcal{C}$ *functors. There are functors* $U : F \downarrow G \to \mathcal{A}$, $V : F \downarrow G \to \mathcal{B}$, *and a natural transformation* $\alpha : F \circ U \Rightarrow G \circ V$, *i.e. the diagram below commutes:*

$$\begin{tikzcd}
F \downarrow G \arrow[r, "U"] \arrow[d, "V"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & \mathcal{A} \arrow[d, "F"] \\
\mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

The functor U is just the projection, i.e. $U\,(A, f, B) = A$ and $U\,(a, b) = a$. V is defined analogously. The components of α are $\alpha_{(A,f,B)} = f$.

As with many constructions in category theory, comma categories come with a universal property. The universal property of comma categories allows to define a unique functor into the comma category from two functors into $\mathcal{A}$ and $\mathcal{B}$, respectively, subject to existence of a natural transformation as detailed below.

Proposition 4.3 (cf. Borceux [2, Proposition 1.6.3]) *In the setting of Proposition 4.2, let* $\mathcal{D}$ *be a further category with functors* $U' : \mathcal{D} \to \mathcal{A}$ *and* $V' : \mathcal{D} \to \mathcal{B}$. *Also, let* $\beta : F \circ U' \Rightarrow G \circ V'$ *be a natural transformation.*

There exists a unique functor $W : \mathcal{D} \to F \downarrow G$ *such that the pasting diagrams below are equal:*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "U'"] \arrow[d, "V'"'] \arrow[dr, phantom, "\overset{\beta}{\swarrow}"] & \mathcal{A} \arrow[d, "F"] \\
\mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{D} \arrow[dr, dashed, "W"] \arrow[drr, bend left, "U'"] \arrow[ddr, bend right, "V'"'] & & \\
& F \downarrow G \arrow[r, "U"] \arrow[d, "V"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & \mathcal{A} \arrow[d, "F"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

The functor W can be defined explicitly as $W\,D := (U'\,D, \beta'_D, V'\,D)$ on objects and $W\,d := (U'\,d, V'\,d)$ on arrows.

In the definition of the fullification of a Banach category $(\mathcal{D}, F, \mathcal{L})$ (Section 4.5), the full functor will be obtained from Proposition 4.3 and the comma category used will have $\mathcal{A} := \mathcal{L}$, $\mathcal{B} := \mathcal{C} := [\mathsf{MCat}(\mathcal{D})^{\mathsf{op}}, \mathsf{Set}]$, where $\mathsf{MCat}(\mathcal{D})$ is a multicategorical construction explained below. The functor G is the identity, and F will be a nerve-like construction.

4.3 Multicategories

In ordinary category, morphisms have a single domain object. In many real-world examples, however, morphisms may take several arguments, think of bilinear maps for instance. The notion of multicategories, due to Lambek [42], addresses this issue and provides a categorical framework to handle morphisms whose domain consists of multiple objects.

The theory of multicategories is vast in its own right, and a multicategory is *not* a special case of a category. However, any multicategory can be transformed into a monoidal category, as observed by Hermida [30, Section 7]. Based on the construction of this transformation, one can define $\mathsf{MCat}(\mathcal{C})$, see below.

The definition of $\mathsf{MCat}(\mathcal{C})$ involves partitions of finite sets. The following notation is used: If P is a partition of $\{1,\dots,n\}$ into m (possibly empty) sets $Q_1,\dots,Q_m$, write $Q_i = \{h(1,i), h(2,i),\dots,h(j_i,i)\} = \{h(k,i) \,|\, k = 1,\dots,j_i\}$. The number $h(k,i)$ is the k-th element of the i-th set in the partition. If $Q_i = \emptyset$, then $j_i = 0$. We then simply write $P = \{h(k,i) \,|\, k = 1,\dots,j_i\}_{i=1}^m$.

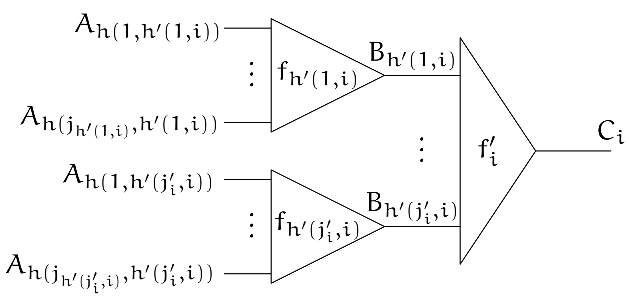
Definition 4.4 *Let $(\mathcal{C},\otimes,I)$ be a monoidal category. The category $\mathsf{MCat}(\mathcal{C})$ is defined as follows:*

- *Objects of $\mathsf{MCat}(\mathcal{C})$ are finite lists of objects in $\mathcal{C}$ (including the empty list).*
- *A morphism $f : [A_1,\dots,A_n] \to [B_1,\dots,B_m]$ is a pair $f = (P,[f_1,\dots,f_m])$, where $P = \{h(k,i) \,|\, k = 1,\dots,j_i\}_{i=1}^m$ is a partition of $\{1,\dots,n\}$ and each f_i is a $\mathcal{C}$-morphism of type*

$$f_i : A_{h(1,i)} \otimes \cdots \otimes A_{h(j_i,i)} \to B_i.$$

  *If $j_i = 0$, then f_i has type $I \to B_i$. The partition P is called the **signature** of f.*

  - *If $f : [A_1,\dots,A_n] \to [B_1,\dots,B_m]$ and $f' : [B_1,\dots,B_m] \to [C_1,\dots,C_\ell]$ are morphisms with signatures $\{h(k,i) \,|\, k = 1,\dots,j_i\}_{i=1}^m$ and $\{h'(k,i) \,|\, k = 1,\dots,j'_i\}_{i=1}^{\ell}$, respectively, then $(f' \circ f)_i$ is defined as in the diagram below:*



– *The identity* $\mathrm{id}_{[A_1,\ldots,A_n]}$ *on* $[A_1,\ldots,A_n]$ *is given by* $[\mathrm{id}_{A_1},\ldots,\mathrm{id}_{A_n}]$.

The composition in $\mathsf{MCat}(\mathcal{C})$ works as follows: The morphism $(f' \circ f)_i$ needs to have codomain C_i. The morphism f'_i also has codomain C_i, so it can be applied, and the new goal is to produce an object of type $B_{h'(1,i)} \otimes \cdots \otimes B_{h'(j'_i,i)}$. Using $f_{h'(1,i)}$, it is possible to obtain $B_{h'(1,i)}$, using $f_{h'(2,i)}$ one gets $B_{h'(2,i)}$ and so forth. This construction ultimately yields the morphism $(f' \circ f)_i := f'_i \circ (f_{h'(1,i)} \otimes \cdots \otimes f_{h'(j'_i,i)})$.

It is helpful to consider a few edge cases of the definition of $\mathsf{MCat}(\mathcal{C})$: There are no $\mathsf{MCat}(\mathcal{C})$-morphisms of type $[A_1,\ldots,A_n] \to []$, where $n > 0$, as $\{1,\ldots,n\}$ cannot be partitioned into 0 sets. An $\mathsf{MCat}(\mathcal{C})$-morphism of type $[] \to [A_1,\ldots,A_n]$ for $n \geqslant 0$ is a list $[f_1,\ldots,f_n]$, where $f_i : I \to A_i$. This follows immediately from the definition.

An arrow between singleton lists $[A]$ and $[B]$ is just a singleton list $[f]$, where $f : A \to B$ is a morphism in $\mathcal{C}$. The identity on $[A]$ is just $[\mathrm{id}_A]$, and $[f] \circ [g] = [f \circ g]$ (all these hold by definition). This is phrased formally in the following proposition.

Proposition 4.5 *Let* $(\mathcal{C}, \otimes, I)$ *be a monoidal category.* $\mathcal{C}$ *is a full subcategory of* $\mathsf{MCat}(\mathcal{C})$ *under the subcategory embedding* $C \mapsto [C], f \mapsto [f]$.

Proof Immediate from the definitions. □

The category $\mathsf{MCat}(\mathcal{C})$ can be endowed with a strict symmetric monoidal structure by simply concatenating the respective lists.

Definition 4.6 *Let* $(\mathcal{C}, \otimes, I)$ *be a monoidal category. Define a strict monoidal structure* $\otimes_{\mathsf{MCat}(\mathcal{C})}$ *on* $\mathsf{MCat}(\mathcal{C})$ *as follows:*

- *On objects,* $[A_1,\ldots,A_n] \otimes_{\mathsf{MCat}(\mathcal{C})} [B_1,\ldots,B_m] := [A_1,\ldots,A_n,B_1,\ldots,B_m]$
- *On arrows,* $[f_1,\ldots,f_n] \otimes_{\mathsf{MCat}(\mathcal{C})} [g_1,\ldots,g_m] := [f_1,\ldots,f_n,g_1,\ldots,g_m]$.

Verifying that this is indeed a (strict) symmetric monoidal structure is routine.

4.4 Day Convolution and Kan Extensions

To prove that the fullification is lax monoidal (Section 4.6), we employ a folklore glueing construction, following Hyland and Schalk's [33, Section 4] development. They define a monoidal structure on certain classes of comma categories. We employ their construction, and for it to be applicable in the setting of this thesis, a monoidal structure on the category $[\mathsf{MCat}(\mathcal{C})^{\mathrm{op}}, \mathsf{Set}]$ is needed, where $\mathcal{C}$ is monoidal.

Day [14] defines a monoidal structure on the presheaves of a given monoidal category. This construction is called the **Day convolution product**. There is also a version for functors $\mathcal{C} \to \mathsf{Set}$, but this is not needed in this thesis. The definition below is taken from Perrone [54, Definition 5.6], with the slight observation that all presheaves on a small category are small [51].

Definition 4.7 (Day Convolution) *Let* $(\mathcal{C}, \otimes, I)$ *be a small monoidal category and let* $P, Q : \mathcal{C}^{op} \to \mathsf{Set}$ *be presheaves on* $\mathcal{C}$. *The* ***Day convolution*** *of* P *and* Q, *written* $P \otimes_{\mathsf{Day}} Q : \mathcal{C}^{op} \to \mathsf{Set}$, *is defined as the following coend:*

$$(P \otimes_{\mathsf{Day}} Q)(X) := \int^{Y, Z \in \mathcal{C}} P\, Y \times Q\, Z \times \mathcal{C}(X, Y \otimes Z)$$

The definition above uses a **coend** (denoted by the integral sign), which are originally due to Yoneda [61]. For this thesis, the definition of coends was not unfolded, but an equivalent formulation of the Day convolution was used that avoids coends, see below. However, to understand coends, the interested reader is referred to the vast literature in the field, for instance the recent textbook account by Loregian [46]. The coend appearing in the definition of Day convolution can be seen as generalisation of the separating conjunction [57, 34] from the logic of bunched implications and separation logic [56].

Proposition 4.8 (cf. Loregian [46, Proposition 6.2.1 and Remark 6.2.4]) *Fix a small (symmetric) monoidal category* $(\mathcal{C}, \otimes, I)$. *Then the tuple* $([\mathcal{C}^{op}, \mathsf{Set}], \otimes_{\mathsf{Day}}, \mathcal{C}(-, I))$ *is a (symmetric) monoidal closed category.*

For the proof of the monoidality of the fullification construction, we need to define arrows in $[\mathcal{C}^{op}, \mathsf{Set}]$ with domain $P \otimes_{\mathsf{Day}} Q$ for some P, Q. This is inconvenient given the intricate definition of the Day convolution. However, it suffices to define an arrow out the Cartesian product of P and Q (details below), which is much easier. The arrow with domain $P \otimes_{\mathsf{Day}} Q$ is then obtained via a universal property. This is achieved via the **Kan extension** characterisation of the Day convolution.

Kan extensions [38] have a rich theory and they concern extending functors in a universal way. Relevant for this thesis are **left Kan extensions**.

Definition 4.9 (Left Kan Extension, cf. Riehl [58, Chapter 6]) *Let* $\mathcal{C}, \mathcal{E}, \mathcal{D}$ *be categories and* $F : \mathcal{C} \to \mathcal{E}, K : \mathcal{C} \to \mathcal{D}$ *functors. A* ***left Kan extension*** *of* F *along* K *is a functor* $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ *alongside a natural transformation* $\eta : F \Rightarrow (\mathrm{Lan}_K F) \circ K$ *such that for all functors* $G : \mathcal{D} \to \mathcal{E}$ *with natural transformation* $\gamma : F \Rightarrow G \circ K$ *there exists a unique natural transformation* $\mathrm{Lan}_K \Rightarrow G$ *making the following pasting diagrams equal:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\gamma\Downarrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\exists!"] & \end{tikzcd}$$

There is also the notion of a right Kan extension, which is exactly like a left Kan extension, with the exception that the natural transformations go in the other direction. For a general account of left (and right) Kan extensions, see e.g. Riehl [58, Chapter 6], who also points out how to compute them.

For a category $\mathcal{C}$ and presheaves $P, Q : \mathcal{C}^{op} \to \mathsf{Set}$, the presheaf $P \mathbin{\overline{\times}} Q : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathsf{Set}$ is defined as $(P \mathbin{\overline{\times}} Q)(X, Y) := P\,X \times Q\,Y$ on objects and $(P \mathbin{\overline{\times}} Q)(f, g) := P\,f \times Q\,g$ on arrows. Day convolution can be phrased as a particular left Kan extension.

Proposition 4.10 (cf. Loregian [46, Proposition 6.2.3]) *Let $(\mathcal{C}, \otimes, I)$ be a small monoidal category and $P, Q : \mathcal{C}^{op} \to \mathsf{Set}$ presheaves on $\mathcal{C}$.*

The presheaf $S := P \otimes_{\mathsf{Day}} Q$ is a left Kan extension of $P \mathbin{\overline{\times}} Q$ along $\otimes$, as illustrated by the pasting diagrams below:

$$
\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \arrow[rr, "P \overline{\times} Q"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\gamma\Downarrow"] & \mathsf{Set} \\
& \mathcal{C}^{op} \arrow[ur, "R"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \arrow[rr, "P \overline{\times} Q"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\eta_{P,Q}\Downarrow"] & \mathsf{Set} \\
& \mathcal{C}^{op} \arrow[ur, "S"] \arrow[ur, bend right, "R"'] \arrow[ur, phantom, "\exists!\,\Downarrow"', near end] &
\end{tikzcd}
$$

The natural transformation $\eta_{P,Q} : P \mathbin{\overline{\times}} Q \Rightarrow P \otimes_{\mathsf{Day}} Q \circ \otimes$ is called the **universal natural transformation** of the Day convolution.

Proposition 4.10 is immensely useful when it comes to proving that the fullification is lax monoidal: Instead of defining natural transformations out of a Day convolution, it suffices to define natural transformations out of Cartesian products, which is much easier.

Furthermore, the operation of $\otimes_{\mathsf{Day}}$ on arrows as well as the structure isos of $\otimes_{\mathsf{Day}}$ can be phrased with this Kan extension characterisation. We were unable to find an explicit construction of the structure isos of $\otimes_{\mathsf{Day}}$ or the operation of $\otimes_{\mathsf{Day}}$ on arrows in the literature, although the Kan extension characterisation of $\otimes_{\mathsf{Day}}$ is well-known. We provide such an explicit construction below.

For the operation of $\otimes_{\mathsf{Day}}$ on arrows, let $P, Q, S, T : \mathcal{C}^{op} \to \mathsf{Set}$ be presheaves and $\alpha : P \Rightarrow Q$, $\beta : S \Rightarrow T$ natural transformations. Then $\alpha \otimes_{\mathsf{Day}} \beta : P \otimes_{\mathsf{Day}} S \Rightarrow S \otimes_{\mathsf{Day}} T$ is defined to be the unique natural transformation such that the following pasting diagrams are equal:

$$
\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \arrow[rr, bend left, "P \overline{\times} S"] \arrow[rr, bend right, "Q \overline{\times} T"'] \arrow[rr, phantom, "\Downarrow\alpha\times\beta"] \arrow[dr, "\otimes"'] & & \mathsf{Set} \\
& \mathcal{C}^{op} \arrow[ur, "Q \otimes_{\mathsf{Day}} T"'] \arrow[u, phantom, "\Downarrow\eta_{Q,T}"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \arrow[rr, "P \overline{\times} S"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\eta_{P,S}\Downarrow"] & \mathsf{Set} \\
& \mathcal{C}^{op} \arrow[ur, bend left, "P \otimes_{\mathsf{Day}} S"] \arrow[ur, bend right, "Q \otimes_{\mathsf{Day}} T"'] \arrow[ur, phantom, "\exists!\,\Downarrow"] &
\end{tikzcd}
$$

Also, the monoidal structure isos of $\otimes_{\mathsf{Day}}$ can be phrased with the help of the Kan extension characterisation. The right unitor $\rho_P : P \otimes_{\mathsf{Day}} \mathcal{C}(-, I_{\mathcal{C}}) \Rightarrow P$, where P :

$\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, is obtained by applying the universal property of $\otimes_{\mathsf{Day}}$ to the natural transformation $\theta : \mathrm{P} \mathbin{\overline{\times}} \mathcal{C}(-, \mathrm{I}_{\mathcal{C}}) \Rightarrow \mathrm{P} \circ \otimes$, which is defined as follows:

$$\theta_{X,Y}(p, f) := (\mathrm{P}\,(\rho_X \circ (\mathrm{id}_X \otimes f)))(p)$$

Note that $\rho_X \circ (\mathrm{id}_X \otimes f)$ has type $X \otimes Y \to X$. Hence, $\mathrm{P}\,(\rho_X \circ (\mathrm{id}_X \otimes f))$ is a function of type $\mathrm{P}\,X \to \mathrm{P}\,(X \otimes Y)$, and since $p \in \mathrm{P}\,X$, we have $(\mathrm{P}\,(\rho_X \circ (\mathrm{id}_X \otimes f)))(p) \in \mathrm{P}\,(X \otimes Y)$. It is routine to check that θ is natural. The right unitor $\rho_{\mathrm{P}} : \mathrm{P} \otimes_{\mathsf{Day}} \mathcal{C}(-, \mathrm{I}_{\mathcal{C}}) \Rightarrow \mathrm{P}$ is then the unique natural transformation making the following pasting diagrams equal, where $\mathrm{S} := \mathrm{P} \otimes_{\mathsf{Day}} \mathcal{C}(-, \mathrm{I}_{\mathcal{C}})$:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[rr, "\mathrm{P} \overline{\times} \mathcal{C}(-,\mathrm{I}_{\mathcal{C}})"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\theta\Downarrow"] & \mathsf{Set} \\ & \mathcal{C}^{\mathrm{op}} \arrow[ur, "\mathrm{P}"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[rr, "\mathrm{P} \overline{\times} \mathcal{C}(-,\mathrm{I}_{\mathcal{C}})"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\eta_{\mathrm{P},\mathcal{C}(-,\mathrm{I}_{\mathcal{C}})}\Downarrow"] & \mathsf{Set} \\ & \mathcal{C}^{\mathrm{op}} \arrow[ur, "\mathrm{S}"] \arrow[ur, bend right, "\mathrm{P}"'] \arrow[ur, phantom, "\exists!\Rightarrow"'] & \end{tikzcd}$$

The left unitor $\lambda_{\mathrm{P}} : \mathcal{C}(-, \mathrm{I}_{\mathcal{C}}) \otimes_{\mathsf{Day}} \mathrm{P} \Rightarrow \mathrm{P}$ can be characterised in a similar fashion. One uses the natural transformation $\mu : \mathcal{C}(-, \mathrm{I}_{\mathcal{C}}) \mathbin{\overline{\times}} \mathrm{P} \Rightarrow \mathrm{P}$ defined as

$$\mu_{X,Y}(f, p) := (\mathrm{P}\,(\lambda_X \circ (f \otimes \mathrm{id}_X)))(p).$$

Also, the associator $\alpha_{\mathrm{P},\mathrm{Q},\mathrm{R}} : (\mathrm{P} \otimes_{\mathsf{Day}} \mathrm{Q}) \otimes_{\mathsf{Day}} \mathrm{R} \Rightarrow \mathrm{P} \otimes_{\mathsf{Day}} (\mathrm{Q} \otimes_{\mathsf{Day}} \mathrm{R})$ can be obtained in this way. One applies the universal property to

$$\nu_{\mathrm{P},\mathrm{Q},\mathrm{R}} : (\mathrm{P} \mathbin{\overline{\times}} \mathrm{Q}) \mathbin{\overline{\times}} \mathrm{R} \Rightarrow \mathrm{P} \mathbin{\overline{\times}} (\mathrm{Q} \mathbin{\overline{\times}} \mathrm{R}) \Rightarrow \mathrm{P} \mathbin{\overline{\times}} ((\mathrm{Q} \otimes_{\mathsf{Day}} \mathrm{R}) \circ \otimes) \Rightarrow (\mathrm{P} \otimes_{\mathsf{Day}} (\mathrm{Q} \otimes_{\mathsf{Day}} \mathrm{R})) \circ \widetilde{\otimes},$$

where ν is defined in the only possible way and $\widetilde{\otimes}(X, Y.Z) := X \otimes (Y \otimes Z)$. Since the transformation ν is indexed by three components, the universal property needs to be applied twice.

The computations above are justified by the standard fact that Kan extensions can be computed as coends (see e.g. Mac Lane [48, Chapter X]), and that the coends corresponding to the structure isos of $\otimes_{\mathsf{Day}}$ as well as the operation of $\otimes_{\mathsf{Day}}$ on morphisms are given by the Kan extensions provided above. Making this formally precise is future work.

4.5 Definition of the Fullification

In the previous sections, all the required preliminaries to define the fullification were introduced. We are therefore now in the position to define it. Since this construction involves several categories and functors, we first give an overview over the entire construction. Then, we explain their roles and provide the proofs. The main result is Corollary 4.17.

For a start, let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathrm{I}_{\mathcal{C}})$, $(\mathcal{L}, \otimes_{\mathcal{L}}, \mathrm{I}_{\mathcal{L}})$ be monoidal categories, and $\mathrm{F} : \mathcal{C} \to \mathcal{L}$ a lax monoidal functor. F is *not* assumed to be full. An overview over the construction is

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathbf{Y}"] \arrow[d, "F"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\beta"] \\
\mathcal{L} \arrow[r, "\mathcal{N}_M(F)"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[dr, dashed, "W"] \arrow[drr, bend left, "\mathbf{Y}"] \arrow[ddr, bend right, "F"'] & & \\
& (\mathrm{id} \downarrow \mathcal{N}_M(F)) \arrow[r, "V"] \arrow[d, "U"] & {[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\alpha"] \\
& \mathcal{L} \arrow[r, "\mathcal{N}_M(F)"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]}
\end{tikzcd}
$$

Figure 4.1: Overview of the fullification construction

given in Figure 4.1. The functor W is what will be the full functor. It will be obtained from the universal property of the comma category (Proposition 4.3). The functors U and V are the projections out of the comma category (see Proposition 4.2). The triangle of F, W, and U is the triangle seen in the chapter introduction. $\mathbf{Y}$ is a version of the Yoneda embedding adapted to $\mathsf{MCat}(\mathcal{C})$ (explained below). Lastly, $\mathcal{N}_M(F)$ is the nerve of F in a version adapted to $\mathsf{MCat}(\mathcal{C})$, also explained below.

The course of action is as follows:

- First, we explain the definitions of $\mathbf{Y}$ and $\mathcal{N}_M(F)$. In particular, it will turn out that $\mathbf{Y}$ is fully faithful, just as the ordinary Yoneda embedding.
- Then, we define a natural transformation β as in the left diagram in Figure 4.1.
- Afterwards, we use the universal property of comma categories to define the functor W.
- Lastly, we verify that W is full, which will rely on fullness of $\mathbf{Y}$.

Definition 4.11 *Let* $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ *be a monoidal category.* $\mathbf{Y} : \mathcal{C} \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ *is the following functor:*

- *On objects,* $\mathbf{Y}\, C := \mathsf{MCat}(\mathcal{C})(-, [C])$.
- *On arrows* $f : C \to D$, $\mathbf{Y}\, f$ *is postcomposition by* $[f] : [C] \to [D]$ (*here,* $[f]$ *is the singleton list whose only element is* f)

Proposition 4.12 $\mathbf{Y}$ *is a fully faithful functor.*

Proof Let $\mathcal{J} : \mathcal{C} \hookrightarrow \mathsf{MCat}(\mathcal{C})$ be the subcategory embedding. By Proposition 4.5, the embedding $\mathcal{J}$ is full and faithful (the latter as it is an embedding). A little calculation shows that $\mathbf{Y} = \mathcal{Y} \circ \mathcal{J}$, where $\mathcal{Y} : \mathsf{MCat}(\mathcal{C}) \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ is the Yoneda embedding. Then $\mathbf{Y}$ is a fully faithful functor as composition of fully faithful functors. $\square$

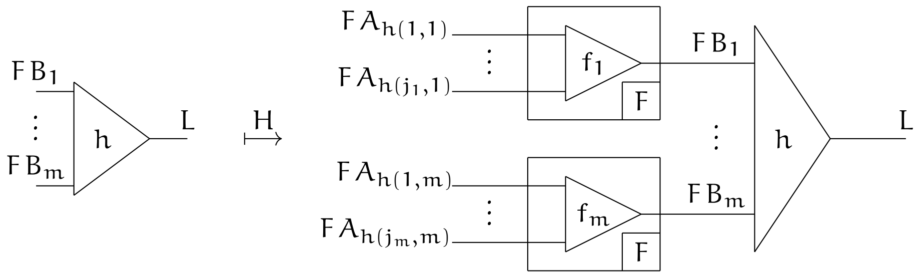
Now that we have an understanding of $\mathbf{Y}$, we draw our attention towards $\mathcal{N}_M(F)$, the multicategorical nerve of F.

Definition 4.13 (Multicategorical Nerve) *Let* $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ *and* $(\mathcal{L}, \otimes_{\mathcal{L}}, I_{\mathcal{L}})$ *be monoidal categories and* $F : \mathcal{C} \to \mathcal{L}$ *lax monoidal. Define a functor* $\mathcal{N}_M(F) : \mathcal{L} \to [\mathsf{MCat}(\mathcal{C})^{op}, \mathsf{Set}]$ *as follows:*

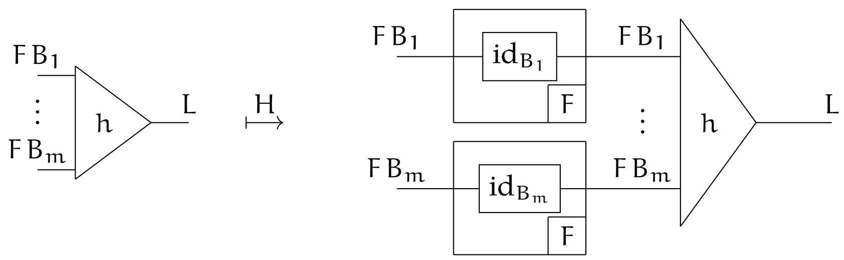
- *On objects,* $\mathcal{N}_M(F)\, L := \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, L)$ *(see below).*
- *On arrows* $g : L \to L'$, $\mathcal{N}_M(F)\, g$ *is postcomposition by* g.

The functor $\mathcal{N}_M(F)$ *is called the* ***multicategorical nerve*** *of* F. *Here, for each* $L \in \mathcal{L}$, *the presheaf* $\mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, L) : \mathsf{MCat}(\mathcal{C})^{op} \to \mathsf{Set}$ *is given as follows:*

- *It sends objects* $[A_1, \dots, A_n]$ *to* $\mathcal{L}(F A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n, L)$. *The empty list is mapped to* $\mathcal{L}(I_{\mathcal{L}}, L)$.
- *It sends a morphism* $f : [A_1, \dots, A_n] \to [B_1, \dots, B_m]$ *to a function* $H : \mathcal{L}(F B_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F B_m, L) \to \mathcal{L}(F A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n, L)$, *where* H *is defined as in the diagram below:*



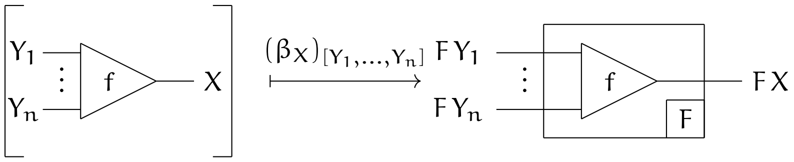
It is clear that $\mathcal{N}_M(F)$ preserves identities and composition, the argument is the same as for the ordinary Yoneda embedding. It is also easy to see that $\mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, L)$ is indeed a presheaf for all $L \in \mathcal{L}$: The identity on a list $[B_1, \dots, B_m]$ is sent to the transformation H which operates as depicted in the following diagram:



Since F is a functor, it is clear that H is the identity transformation. $\mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, L)$ also respects composition. This is straightforward to figure out using the definition of composition in $\mathsf{MCat}(\mathcal{C})$, but extremely tedious to write down formally.

We now have all the required terminology to construct the fullification. The first step is to define a natural transformation $\beta : \mathbf{Y} \Rightarrow \mathcal{N}_M(F) \circ F$ (see Figure 4.1). This is relatively straightforward.

$$\beta_X : \mathbf{Y}\,X = \mathsf{MCat}(\mathcal{C})(-,[X]) \Rightarrow (\mathcal{N}_M(F) \circ F)\,X = \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, F\,X)$$
$$(\beta_X)_{[Y_1,\ldots,Y_n]} : \mathsf{MCat}(\mathcal{C})([Y_1,\ldots,Y_n],[X]) \to \mathcal{L}(F\,Y_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,Y_n, F\,X)$$



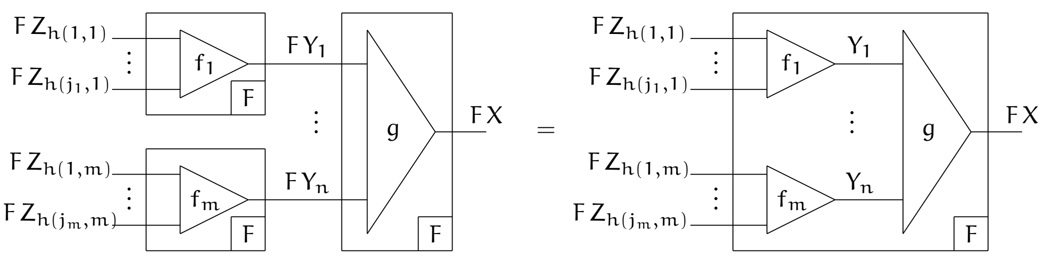
Recall that morphisms in $\mathsf{MCat}(\mathcal{C})$ of type $[Y_1,\ldots,Y_n] \to [X]$ are singleton lists, which is reflected in the diagram above by using the brackets. It turns out that β is well defined (i.e. for all X, β_X is a natural transformation), and that β_X itself is natural in X. This is subject to the subsequent two lemmas.

Lemma 4.14 *β is well-defined, i.e. for all X, β_X is a natural transformation.*

Proof We show that for all $X \in \mathcal{C}$, the mapping β_X is a natural transformation. This is subject to the following naturality square, where $\mathbb{Y} := [Y_1,\ldots,Y_n]$, $\mathbb{Z} := [Z_1,\ldots,Z_m]$, and $[f_1,\ldots,f_n] : \mathbb{Z} \to \mathbb{Y}$.

$$\begin{tikzcd}
{[g]} \arrow[rr, mapsto] \arrow[dd, bend right=60, mapsto] & & F\,[g] \arrow[dd, bend left=60, mapsto] \\
\mathsf{MCat}(\mathcal{C})(\mathbb{Y},[X]) \arrow[r, "(\beta_X)_{\mathbb{Y}}"] \arrow[d, "-\circ[f_1,\ldots,f_n]"'] & \mathcal{L}(F\,Y_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,Y_n, F\,X) \arrow[d, "H"] & \\
\mathsf{MCat}(\mathcal{C})(\mathbb{Z},[X]) \arrow[r, "(\beta_X)_{\mathbb{Z}}"'] & \mathcal{L}(F\,Z_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,Z_m, F\,X) & \\
{[g] \circ [f_1,\ldots,f_n]} \arrow[rr, mapsto] & & F\,([g] \circ [f_1,\ldots,f_n]) = H\,(F\,[g])
\end{tikzcd}$$

To prove that the square commutes, it suffices to verify that $H\,(F\,[g]) = F\,([g] \circ [f_1,\ldots,f_n])$. Let $\{h(k,i) \mid k = 1,\ldots,j_i\}_{i=1}^{m}$ be the signature of $[f_1,\ldots,f_n]$. The equation $H\,(F\,[g]) = F\,([g] \circ [f_1,\ldots,f_n])$ is depicted below:



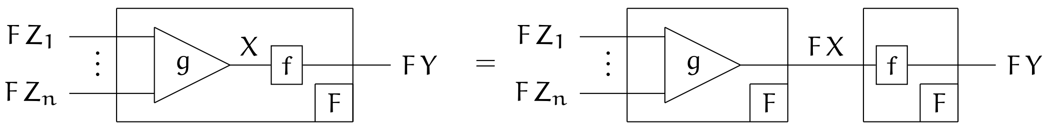
Both sides are equal by the last diagram equality given in Section 2.2. □

Lemma 4.15 *β is a natural transformation.*

Proof It needs to be shown that β_X is natural in X, i.e. that the following square commutes, where $X, Y \in \mathcal{C}$ and $f : X \to Y$.

$$\begin{tikzcd}
\mathbf{Y}X \arrow[r, "\beta_X"] \arrow[d, "{[f]\circ-}"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, FX) \arrow[d, "(Ff)\circ-"] \\
\mathbf{Y}Y \arrow[r, "\beta_Y"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, FY)
\end{tikzcd}$$

To prove this, let $\mathbb{Z} = [Z_1, \ldots, Z_n]$ be an object of $\mathsf{MCat}(\mathcal{C})$, and fix $[g] : \mathbb{Z} \to [X]$. It suffices to show $(\beta_Y)_{\mathbb{Z}}([f] \circ [g]) = (Ff) \circ (\beta_X)_{\mathbb{Z}}([g])$. In diagrams, after applying the definition of β and noting that $[f \circ g] = [f] \circ [g]$, this equality looks as follows:



Both sides are equal since functors preserve composition. $\square$

We now know that $\beta : \mathbf{Y} \Rightarrow \mathcal{N}_M(F) \circ F$ is natural. Consider again the diagram in Figure 4.1. Ultimately, the universal property of the comma category (Proposition 4.3) is to be applied to obtain the functor W, which will be the candidate for the fullification. It turns out that we are already in the position to do so: Applying Proposition 4.3 to F, $\mathbf{Y}$, and β yields a functor $W : \mathcal{C} \to \mathrm{id} \downarrow \mathcal{N}_M(F)$ such that $F = W \circ U$, $\mathbf{Y} = V \circ W$, and $\beta = \alpha * W$.

Explicitly, W is defined as follows: $WX = (\mathsf{MCat}(\mathcal{C})(-, [X]), \beta_X, FX)$, where $X \in \mathcal{C}$, and $Wg = (\mathbf{Y}g, Fg)$ for $g : X \to Y$ and $X, Y \in \mathcal{C}$. The final step of the argument is to prove that W is full. The proof below mostly relies on the Yoneda lemma.

Theorem 4.16 *The functor W is full.*

Proof Let $X, Y \in \mathcal{C}$ and $f : WX \to WY$. We need to find $g : X \to Y$ such that $Wg = f$.

By definition, $WX = (\mathsf{MCat}(\mathcal{C})(-, [X]), \beta_X, FX)$ and $WY = (\mathsf{MCat}(\mathcal{C})(-, [Y]), \beta_Y, FY)$. Since f is an arrow in the comma category, there exist

$$a : \mathsf{MCat}(\mathcal{C})(-, [X]) \Rightarrow \mathsf{MCat}(\mathcal{C})(-, [Y])$$

and $b : FX \to FY$ such that $f = (a, b)$. As $\mathbf{Y}$ is full by Proposition 4.12, there exists $g : X \to Y$ such that $\mathbf{Y}g = a$. To conclude the proof, it suffices to show that $Fg = b$ since then $Wg = (\mathbf{Y}g, Fg) = (a, b) = f$, which was claimed.

As $(\mathbf{Y}\, g, b) = (a, b) = f$ is a morphism in the comma category, the following square commutes:

$$\begin{tikzcd}
\mathsf{MCat}(\mathcal{C})(-, [X]) \arrow[r, "\beta_X"] \arrow[d, "{[g]\circ -}"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F -, F X) \arrow[d, "b \circ -"] \\
\mathsf{MCat}(\mathcal{C})(-, [Y]) \arrow[r, "\beta_Y"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F -, F Y)
\end{tikzcd}$$

In particular, this implies commutativity of

$$\begin{tikzcd}
\mathsf{MCat}(\mathcal{C})([X], [X]) \arrow[r, "(\beta_X)_{[X]}"] \arrow[d, "{[g]\circ -}"'] & \mathcal{L}(F X, F X) \arrow[d, "b \circ -"] \\
\mathsf{MCat}(\mathcal{C})([X], [Y]) \arrow[r, "(\beta_Y)_{[X]}"'] & \mathcal{L}(F X, F Y)
\end{tikzcd}$$

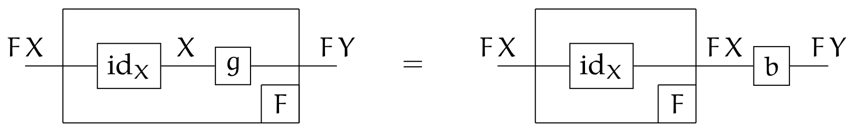
Observe that trivially $[\mathrm{id}_X] \in \mathsf{MCat}(\mathcal{C})([X], [X])$. Commutativity of the square above then implies that

$$(\beta_Y)_{[X]}([g] \circ [\mathrm{id}_X]) = b \circ (\beta_X)_{[X]}([\mathrm{id}_X]).$$

Since $[g] \circ [\mathrm{id}_X] = [g \circ \mathrm{id}_X]$, we also have the equality

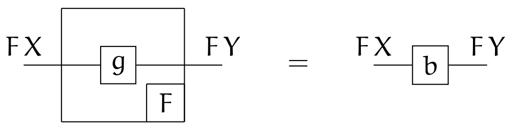
$$(\beta_Y)_{[X]}([g \circ \mathrm{id}_X]) = b \circ (\beta_X)_{[X]}([\mathrm{id}_X]). \tag{4.1}$$

After unfolding the definition of β, Equation (4.1) can be depicted as follows:



(4.2)

Since F is a functor, it respects identities and composition, so Equation (4.2) can be reduced to:



This is to say that $F\, g = b$, as claimed. $\square$

The development of this section is succinctly summarised in the following corollary.

Corollary 4.17 *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and $(\mathcal{L}, \otimes_{\mathcal{L}}, I_{\mathcal{L}})$ be monoidal categories. Further, let $F : \mathcal{C} \to \mathcal{L}$ be lax monoidal. Then, there exists a category $\widetilde{\mathcal{L}}$, a full functor $\widetilde{F} : \mathcal{C} \to \widetilde{\mathcal{L}}$, and a functor $q : \widetilde{\mathcal{L}} \to \mathcal{L}$ such that the following triangle commutes:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & \widetilde{\mathcal{L}} \arrow[d, "q"] \\
& \mathcal{L}
\end{tikzcd}$$

Proof Set $\widetilde{\mathcal{L}} := \mathrm{id} \downarrow \mathcal{N}_M(F)$. Let $q : \mathrm{id} \downarrow \mathcal{N}_M(F) \to \mathcal{L}$ be the projection out of the comma category (see Proposition 4.2). Let $\widetilde{F} : \mathcal{C} \to \mathrm{id} \downarrow \mathcal{N}_M(F)$ be the functor defined using the universal property of the comma category applied to $\mathbf{Y}$, F, and β. The equality $F = q \circ \widetilde{F}$ holds by definition, and fullness is Theorem 4.16. □

4.6 Lax Monoidal Structure of the Fullification

In the previous section, a lax monoidal functor between two monoidal categories was fullified, but we did not yet verify that the fullification is again a lax monoidal functor. In the following we prove that the fullification of a Banach category $(\mathcal{C}, F, \mathcal{L})$ is a Banach category as well, assuming that $\mathcal{C}$ is small.

The high-level architecture of our argument is as follows (for reference, consult again Figure 4.1):

- We review a method to give a monoidal structure on the comma category $\mathrm{id} \downarrow \mathcal{N}_M(F)$. It is called "glueing" and the construction is folklore. In the present thesis, we follow the development of Hyland and Schalk [33, Section 4] who describe glueing.
- We instantiate the glueing construction to the setting depicted in Figure 4.1. This will involve proving that $\mathcal{N}_M(F) : \mathcal{L} \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ is lax monoidal (Lemma 4.24), which is difficult because the monoidal structure on the presheaves in $[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ is given by the relatively technical Day convolution.
- Using the monoidal structure of $\mathrm{id} \downarrow \mathcal{N}_M(F)$, we prove that the full functor W is lax monoidal as well (Theorem 4.26). This is still technical, but slightly easier than proving lax monoidality of $\mathcal{N}_M(F)$.
- Finally, we wrap everything up and prove the fullification result in Corollary 4.28.

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}), (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ be symmetric monoidal categories, and $G : \mathcal{C} \to \mathcal{D}$ a lax monoidal functor with coherence morphisms ε and $\mu_{A,B}$. Consider the comma category $\mathrm{id} \downarrow G$ together with the projections V and U:

$$\begin{tikzcd}
\mathrm{id} \downarrow G \arrow[r, "V"] \arrow[d, "U"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{D} \arrow[d, "\mathrm{id}"] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}$$

It turns out that $\mathrm{id} \downarrow G$ has a canonical symmetric monoidal structure.

Lemma 4.18 (cf. Hyland and Schalk [33, Proposition 25]) $\mathrm{id} \downarrow G$ *admits a symmetric monoidal structure given by*

$$(A, f, B) \otimes (A', f', B') := (A \otimes_{\mathcal{D}} A', \mu_{B,B'} \circ (f \otimes_{\mathcal{D}} f'), B \otimes_{\mathcal{C}} B')$$

on objects and $(a, b) \otimes (a', b') := (a \otimes_{\mathcal{D}} a', b \otimes_{\mathcal{C}} b')$ *on arrows. The monoidal unit is* $(I_{\mathcal{D}}, \varepsilon, I_{\mathcal{C}})$.

Lemma 4.19 (cf. Hyland and Schalk [33, Proposition 25]) *The projections* $V : \mathrm{id} \downarrow G \to \mathcal{D}$ *and* $U : \mathrm{id} \downarrow G \to \mathcal{C}$ *are strict monoidal with respect to the monoidal structure given in Lemma 4.18.*

Even stronger, $\mathrm{id} \downarrow G$ is also monoidal closed, subject to certain conditions.

Lemma 4.20 (cf. Hyland and Schalk [33, Proposition 26]) *Let further* $\mathcal{C}, \mathcal{D}$ *have a closed symmetric monoidal structure and let* $\mathcal{D}$ *have pullbacks. Then* $\mathrm{id} \downarrow G$ *is symmetric monoidal closed with respect to the monoidal structure from Lemma 4.18. Further,* U *and* V *are functors between SMCCs as in Definition 2.21.*

This construction will now be applied to Figure 4.1. The relevant part for the glueing construction is the following subdiagram of Figure 4.1:

$$\begin{tikzcd}
 & (\mathrm{id} \downarrow \mathcal{N}_M(F)) \arrow[r, "V"] \arrow[d, "U"'] \arrow[dr, phantom, "\Leftarrow\alpha"] & {[\mathsf{MCat}(\mathcal{C})^{op}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{L} \arrow[r, "\mathcal{N}_M(F)"'] & {[\mathsf{MCat}(\mathcal{C})^{op}, \mathsf{Set}]}
\end{tikzcd}$$

From now on, we will freely use the notation from the above diagram. In addition to the setting of Section 4.5, we suppose that $\mathcal{L}$ is symmetric monoidal closed. The SMCC-structure on $[\mathsf{MCat}(\mathcal{C})^{op}, \mathsf{Set}]$ is given by the Day convolution (Definition 4.7). Notice that since $\mathcal{C}$ is small, then so is $\mathsf{MCat}(\mathcal{C})$.

To make Lemmas 4.19 and 4.20 applicable, the following things need to be established: Pullbacks in $[\mathsf{MCat}(\mathcal{C})^{op}, \mathsf{Set}]$ and lax monoidality of the nerve $\mathcal{N}_M(F)$. The former is standard, the latter the meat of this section.

It is a standard result in category theory that limits of presheaves over small categories are computed pointwise, see for instance the textbook account by Borceux [2, Corollary 2.15.4]. Since pullbacks are limits of a particular shape, $[\mathsf{MCat}(\mathcal{C})^{op}, \mathsf{Set}]$ has pullbacks.

The more intricate part is proving lax monoidality of $\mathcal{N}_M(F)$. This boils down to defining arrows $\varepsilon : \mathsf{MCat}(\mathcal{C})(-, []) \Rightarrow \mathcal{N}_M(F)\, I_{\mathcal{L}}$ and $\mu_{X,Y} : \mathcal{N}_M(F)\, X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\, Y \Rightarrow \mathcal{N}_M(F)\, (X \otimes_{\mathcal{L}} Y)$, natural in X and Y.

It is straightforward to define ε: For all nonempty lists $[A_1, \ldots, A_n] =: \mathbb{A}$, the hom-set $\mathsf{MCat}(\mathcal{C})(\mathbb{A}, [])$ is empty, so the component $\varepsilon_{\mathbb{A}} : \mathsf{MCat}(\mathcal{C})(\mathbb{A}, []) \to \mathcal{L}(F A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n, I_{\mathcal{L}})$ is trivial to define. Furthermore, the hom-set $\mathsf{MCat}(\mathcal{C})([], [])$ is a singleton, and its only element is the identity on $[]$. Define $\varepsilon_{[]}(\mathrm{id}_{[]}) := \mathrm{id}_{I_{\mathcal{L}}}$ (recall that $(\mathcal{N}_M(F)\, I_{\mathcal{L}})\, [] = \mathcal{L}(I_{\mathcal{L}}, I_{\mathcal{L}})$). Naturality of ε is almost trivial. The only nontrivial case for the naturality square is to show that for all $n > 0$ and $f : [] \to [A_1, \ldots, A_n]$ the following diagram commutes:

$$\begin{tikzcd}
\mathsf{MCat}(\mathcal{C})([], []) \arrow[r, "\varepsilon_{[]}"] \arrow[d, "\mathsf{MCat}(\mathcal{C})(-,[])\, f"'] & \mathcal{L}(I_{\mathcal{L}}, I_{\mathcal{L}}) \arrow[d, "\mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F -, I_{\mathcal{L}})\, f"] \\
\mathsf{MCat}(\mathcal{C})([A_1, \ldots, A_n], []) \arrow[r, "\varepsilon_{[A_1,\ldots,A_n]}"'] & \mathcal{L}(F A_1, \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n, I_{\mathcal{L}}))
\end{tikzcd}$$

But since there are no morphisms of type $[] \to [A_1, \ldots, A_n]$, it follows that ε is indeed natural.

The definition of μ, however, poses a hurdle since it involves Day convolution, which is defined in terms of coends, a concept not touched in this thesis. To mitigate this issue, we employ the Kan extension characterisation of the Day convolution (Proposition 4.10). Using this fact, it suffices to define a natural transformation $\gamma_{X,Y} : \mathcal{N}_M(F)\, X \,\overline{\times}\, \mathcal{N}_M(F)\, Y \Rightarrow \mathcal{N}_M(F)\, (X \otimes_{\mathcal{L}} Y) \circ \otimes_{\mathsf{MCat}(\mathcal{C})}$, natural in X and Y, which is much simpler (justification follows). Note that γ will be a natural transformation whose components are again natural transformations.

Given $\mathbb{A} := [A_1, \ldots, A_n]$ and $\mathbb{B} := [B_1, \ldots, B_m]$ in $\mathsf{MCat}(\mathcal{C})$, define $(\gamma_{X,Y})_{\mathbb{A},\mathbb{B}}$ as follows:

$$\begin{aligned}
(\gamma_{X,Y})_{\mathbb{A},\mathbb{B}} : \mathcal{L}(F A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n, X) \times \mathcal{L}(F B_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F B_m, Y)& \\
\to \mathcal{L}(F A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n \otimes_{\mathcal{L}} F B_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F B_m, X \otimes_{\mathcal{L}} Y)& \\
(f, g) \mapsto f \otimes_{\mathcal{L}} g&
\end{aligned}$$

As usual, the notation $f \otimes_{\mathcal{L}} g$ is to be understood up to composition with the structure isos of $\mathcal{L}$. The subsequent two lemmas assert that γ is well-defined (i.e. for all X, Y, $\gamma_{X,Y}$ is a natural transformation), and $\gamma_{X,Y}$ itself is natural in X, Y.

Lemma 4.21 *γ is well-defined, i.e. for all X, Y, $\gamma_{X,Y}$ is a natural transformation.*

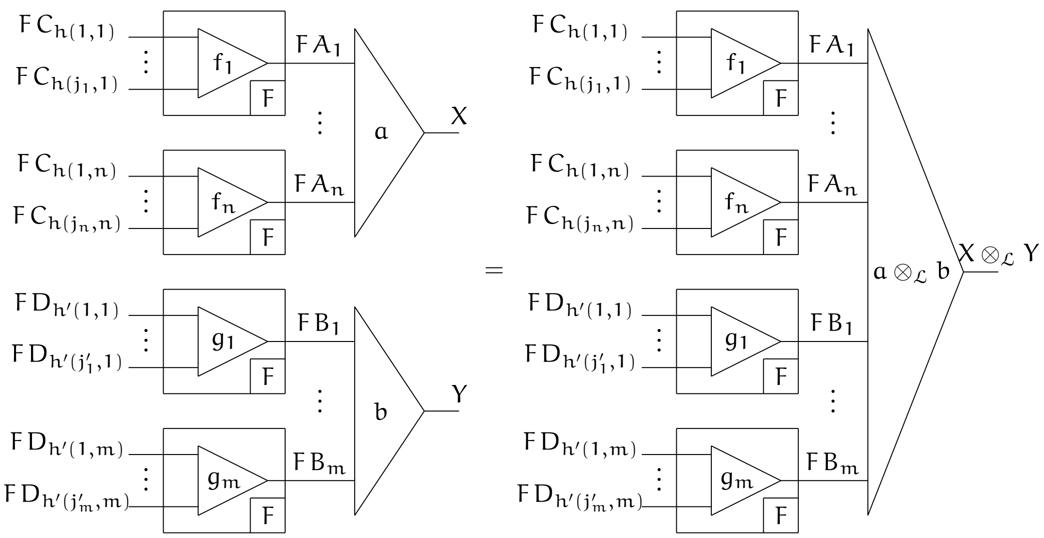
Proof Let $\mathbb{A} := [A_1, \ldots, A_n]$, $\mathbb{B} := [B_1, \ldots, B_m]$, $\mathbb{C} := [C_1, \ldots, C_k]$, as well as $\mathbb{D} := [D_1, \ldots, D_\ell]$ be $\mathsf{MCat}(\mathcal{C})$-objects. Further, let $[f_1, \ldots, f_n] : \mathbb{C} \to \mathbb{A}$ and $[g_1, \ldots, g_m] : \mathbb{D} \to \mathbb{B}$ be morphisms. Naturality of $\gamma_{X,Y}$ is subject to the following square, where $F\,\mathbb{A}$ denotes $F A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F A_n$:

$$\begin{tikzcd}
\mathcal{L}(F\,\mathbb{A}, X) \times \mathcal{L}(F\,\mathbb{B}, Y) \arrow[r, "(\gamma_{X,Y})_{\mathbb{A},\mathbb{B}}"] \arrow[d, "\mathcal{L}([f_1,\ldots,f_n],X) \times \mathcal{L}([g_1,\ldots,g_m],Y)"'] & \mathcal{L}(F\,\mathbb{A} \otimes_{\mathcal{L}} F\,\mathbb{B}, X \otimes_{\mathcal{L}} Y) \arrow[d, "\mathcal{L}([f_1,\ldots,f_n,g_1,\ldots,g_m], X \otimes_{\mathcal{L}} Y)"] \\
\mathcal{L}(F\,\mathbb{C}, X) \times \mathcal{L}(F\,\mathbb{D}, Y) \arrow[r, "(\gamma_{X,Y})_{\mathbb{C},\mathbb{D}}"] & \mathcal{L}(F\,\mathbb{C} \otimes_{\mathcal{L}} F\,\mathbb{D}, X \otimes_{\mathcal{L}} Y)
\end{tikzcd}$$

Recall that by definition, it holds that

$$[f_1, \dots, f_n] \otimes_{\mathsf{MCat}(\mathcal{C})} [g_1, \dots, g_m] = [f_1, \dots, f_n, g_1, \dots, g_m].$$

To prove commutativity of the square above, let $(a, b) \in \mathcal{L}(F\,\mathbb{A}, X) \times \mathcal{L}(F\,\mathbb{B}, Y)$. The path along the left and bottom arrows of the square is depicted on the left hand side of the following diagram, the one along the top and right arrows on the right side of the following diagram.



Both sides are equal *by definition* of string diagrams. That is, $\gamma_{X,Y}$ is natural. $\square$

Lemma 4.22 *γ is a natural transformation.*

Proof Let X, Y, W, Z be $\mathcal{L}$-objects and $f : X \to W$, $g : Y \to Z$ be $\mathcal{L}$-morphisms. To prove that γ is a natural transformation, the square below needs to commute:

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \,\overline{\times}\, \mathcal{N}_M(F)\,Y \arrow[r, "\gamma_{X,Y}"] \arrow[d, "f\circ- \times g\circ-"'] & \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} Y) \circ \otimes_{\mathsf{MCat}(\mathcal{C})} \arrow[d, "(f \otimes_{\mathcal{L}} g)\circ-"] \\
\mathcal{N}_M(F)\,W \,\overline{\times}\, \mathcal{N}_M(F)\,Z \arrow[r, "\gamma_{W,Z}"] & \mathcal{N}_M(F)\,(W \otimes_{\mathcal{L}} Z) \circ \otimes_{\mathsf{MCat}(\mathcal{C})}
\end{tikzcd}$$

To prove that this square commutes, let $\mathbb{A} = [A_1, \dots, A_n]$, $\mathbb{B} := [B_1, \dots, B_m]$ be $\mathsf{MCat}(\mathcal{C})$-objects. It suffices to show that the square below commutes, which is trivial by functoriality of $\otimes_{\mathcal{L}}$. $\square$

$$\begin{tikzcd}
(a,b) \arrow[rrr, mapsto] \arrow[d, bend right] & & & a \otimes_{\mathcal{L}} b \arrow[d, bend left] \\
\mathcal{L}(F\mathbb{A},X) \times \mathcal{L}(F\mathbb{B},Y) \arrow[rrr, "(\gamma_{X,Y})_{\mathbb{A},\mathbb{B}}"] \arrow[d, "(f\circ -)\times(g\circ -)"'] & & & \mathcal{L}(F\mathbb{A}\otimes_{\mathcal{L}} F\mathbb{B}, X\otimes_{\mathcal{L}} Y) \arrow[d, "(f\otimes_{\mathcal{L}} g)\circ -"] \\
\mathcal{L}(F\mathbb{A},W) \times \mathcal{L}(F\mathbb{B},Z) \arrow[rrr, "(\gamma_{W,Z})_{\mathbb{A},\mathbb{B}}"] & & & \mathcal{L}(F\mathbb{A}\otimes_{\mathcal{L}} F\mathbb{B}, W\otimes_{\mathcal{L}} Z) \\
(f\circ a, g\circ b) \arrow[rrr, mapsto] & & & (f\circ a)\otimes_{\mathcal{L}} (g\circ b) = (f\otimes_{\mathcal{L}} g)\circ(a\otimes_{\mathcal{L}} b)
\end{tikzcd}$$

We are now in the position to apply Proposition 4.10. For all $X, Y \in \mathcal{L}$, there exists a unique natural transformation $\mu_{X,Y} : \mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Y \Rightarrow \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} Y)$ such that $\gamma_{X,Y} = (\mu_{X,Y} * \otimes_{\mathsf{MCat}(\mathcal{C})}) \circ \xi_{X,Y}$, where $\xi_{X,Y} : \mathcal{N}_M(F)\,X \,\overline{\times}\, \mathcal{N}_M(F)\,Y \Rightarrow (\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Y) \circ \otimes_{\mathsf{MCat}(\mathcal{C})}$ is the universal natural transformation from Proposition 4.10.

It is, however, not yet clear that $\mu_{X,Y}$ is natural in X and Y, which is required to qualify as a coherence morphism. Proving this naturality is subject to the next lemma. After that, we will verify the coherence conditions in Lemma 4.24, which will finish the proof that $\mathcal{N}_M(F)$ is lax monoidal.

Lemma 4.23 *$\mu_{X,Y}$ is natural in X and Y.*

Proof Let $X, Y, W, Z \in \mathcal{L}$ and fix $f : X \to W$ as well as $g : Y \to Z$. Naturality of μ is subject to the following square:

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Y \arrow[r, "\mu_{X,Y}"] \arrow[d, "(f\circ -)\otimes_{\mathsf{Day}}(g\circ -)"'] & \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} Y) \arrow[d, "(f\otimes_{\mathcal{L}} g)\circ -"] \\
\mathcal{N}_M(F)\,W \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Z \arrow[r, "\mu_{W,Z}"'] & \mathcal{N}_M(F)\,(W \otimes_{\mathcal{L}} Z)
\end{tikzcd}$$

To prove that the square above commutes, it is helpful to recall how $(f\circ -)\otimes_{\mathsf{Day}}(g\circ -)$ is defined. The natural transformation $(f \circ -) \otimes_{\mathsf{Day}} (g \circ -)$ is the unique natural transformation making the following diagram commute (abbreviating $\otimes_{\mathsf{MCat}(\mathcal{C})}$ as $\otimes$):

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \,\overline{\times}\, \mathcal{N}_M(F)\,Y \arrow[r, "\xi_{X,Y}"] \arrow[d, "(f\circ -)\times(g\circ -)"'] & (\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Y) \circ \otimes \arrow[d, "(f\circ -)\otimes_{\mathsf{Day}}(g\circ -)*\otimes"] \\
\mathcal{N}_M(F)\,W \,\overline{\times}\, \mathcal{N}_M(F)\,Z \arrow[r, "\xi_{W,Z}"'] & (\mathcal{N}_M(F)\,W \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Z) \circ \otimes
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \,\overline{\times}\, \mathcal{N}_M(F)\,Y \arrow[r, "\xi_{X,Y}"] \arrow[d, "(f\circ-)\times(g\circ-)"'] & (\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Y) \circ \otimes \arrow[r, "\mu_{X,Y} * \otimes"] \arrow[d, "(f\circ-)\otimes_{\mathsf{Day}}(g\circ-)*\otimes"] & \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} Y) \circ \otimes \arrow[d, "((f\otimes_{\mathcal{L}} g)\circ-)*\otimes"] \\
\mathcal{N}_M(F)\,W \,\overline{\times}\, \mathcal{N}_M(F)\,Z \arrow[r, "\xi_{W,Z}"'] & (\mathcal{N}_M(F)\,W \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Z) \circ \otimes \arrow[r, "\mu_{W,Z} * \otimes"'] & \mathcal{N}_M(F)\,(W \otimes_{\mathcal{L}} Z) \circ \otimes
\end{tikzcd}$$

The left square commutes by definition of $(f \circ -) \otimes_{\mathsf{Day}} (g \circ -)$. Also, the outer rectangle commutes by the definition of μ (since $\gamma_{X,Y} = (\mu_{X,Y} * \otimes) \circ \xi_{X,Y}$ and $\gamma_{W,Z} = (\mu_{W,Z} * \otimes) \circ \xi_{W,Z}$) and the fact that $\gamma_{X,Y}$ is natural in X, Y by Lemma 4.22. Now, observe the following: The paths along the top and then down as well as the path along $\xi_{X,Y}$, then down, and then right are both equal. This follows from the previous observations by just chasing arrows.

Formally, this is to say that the following natural transformations are equal:

$$\begin{aligned}
&(((f \otimes_{\mathcal{L}} g) \circ -) * \otimes) \circ (\mu_{X,Y} * \otimes) \circ \xi_{X,Y} \\
&= (\mu_{W,Z} * \otimes) \circ ((f \circ -) \otimes_{\mathsf{Day}} (g \circ -) * \otimes) \circ \xi_{X,Y}
\end{aligned}$$

Using that whiskering and composition of natural transformations commute, one obtains the following equality:

$$((\mu_{X,Y} \circ ((f \otimes_{\mathcal{L}} g) \circ -)) * \otimes) \circ \xi_{X,Y} = ((\mu_{W,Z} \circ ((f \circ -) \otimes_{\mathsf{Day}} (g \circ -))) * \otimes) \circ \xi_{X,Y}$$

By the Kan extension characterisation of the Day convolution (Proposition 4.10, the uniqueness part), one obtains

$$\mu_{X,Y} \circ ((f \otimes_{\mathcal{L}} g) \circ -) = \mu_{W,Z} \circ ((f \circ -) \otimes_{\mathsf{Day}} (g \circ -)),$$

i.e. that the desired naturality square for μ (see the beginning of the proof) commutes. That is, $\mu_{X,Y}$ is natural in X, Y, as claimed. □

The current goal is to prove that $\mathcal{N}_M(F)$ is lax monoidal. So far, we have defined arrows $\varepsilon : \mathsf{MCat}(\mathcal{C})(-, []) \Rightarrow \mathcal{N}_M(F)\, I_{\mathcal{L}}$ and $\mu_{X,Y} : \mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,Y \Rightarrow \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} Y)$, where $\mu_{X,Y}$ is natural in X and Y. To show that $\mathcal{N}_M(F)$ is lax monoidal, it now suffices to verify the coherence diagrams for ε and $\mu_{X,Y}$ (see Definition 2.22). This is subject to the next lemma. In its proof, the Kan extension characterisation of the Day convolution is again very useful.

Lemma 4.24 $\mathcal{N}_M(F)$ *is lax monoidal.*

Proof It suffices to verify that ε and $\mu_{X,Y}$ satisfy the coherence conditions.

We start with the left coherence diagram, which is as follows, where $X \in \mathcal{C}$:

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathsf{MCat}(\mathcal{C})(-,\square) \arrow[r, "\rho_{\mathcal{N}_M(F)\,X}"] \arrow[d, "\mathrm{id}_{\mathcal{N}_M(F)\,X} \otimes_{\mathsf{Day}} \varepsilon"'] & \mathcal{N}_M(F)\,X \\
\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,I_{\mathcal{L}} \arrow[r, "\mu_{X,I_{\mathcal{L}}}"'] & \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} I_{\mathcal{L}}) \arrow[u, "\rho_X \circ -"']
\end{tikzcd} \tag{4.3}$$

Now, by definition, $\mathrm{id}_{\mathcal{N}_M(F)\,X} \otimes_{\mathsf{Day}} \varepsilon$ is unique such that the following diagram commutes (abbreviating $\otimes_{\mathsf{MCat}(\mathcal{C})}$ as $\otimes$):

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \overline{\times} \mathsf{MCat}(\mathcal{C})(-,\square) \arrow[r, "\eta"] \arrow[d, "\mathrm{id}_{\mathcal{N}_M(F)\,X} \times \varepsilon"] & (\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathsf{MCat}(\mathcal{C})(-,\square)) \circ \otimes \arrow[d, "(\mathrm{id}_{\mathcal{N}_M(F)\,X} \otimes_{\mathsf{Day}} \varepsilon) * \otimes"] \\
\mathcal{N}_M(F)\,X \overline{\times} \mathcal{N}_M(F)\,I_{\mathcal{L}} \arrow[r, "\xi_{X,I_{\mathcal{L}}}"'] & (\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\,I_{\mathcal{L}}) \circ \otimes
\end{tikzcd} \tag{4.4}$$

Here, η and $\xi_{X,I_{\mathcal{L}}}$ are the universal natural transformations as in Proposition 4.10.

Similarly, $\rho_{\mathcal{N}_M(F)\,X}$ is the unique natural transformation such that the following diagram commutes, where $\theta_{\mathcal{N}_M(F)\,X}$ is as in the paragraph below Proposition 4.10:

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \overline{\times} \mathsf{MCat}(\mathcal{C})(-,\square) \arrow[r, "\eta"] \arrow[dr, "\theta_{\mathcal{N}_M(F)\,X}"'] & (\mathcal{N}_M(F)\,X \otimes_{\mathsf{Day}} \mathsf{MCat}(\mathcal{C})(-,\square)) \circ \otimes \arrow[d, "\rho_{\mathcal{N}_M(F)\,X} * \otimes"] \\
& \mathcal{N}_M(F)\,X \circ \otimes
\end{tikzcd} \tag{4.5}$$

Our goal is to show that Diagram 4.3 commutes. By the Kan extension characterisation of $\otimes_{\mathsf{Day}}$, it suffices to show that

$$(\rho_{\mathcal{N}_M(F)\,X} * \otimes) \circ \eta = ((\rho_X \circ -) * \otimes) \circ (\mu_{X,I_{\mathcal{L}}} * \otimes) \circ ((\mathrm{id}_{\mathcal{N}_M(F)\,X} \otimes_{\mathsf{Day}} \varepsilon) * \otimes) \circ \eta.$$

Now, using Diagrams 4.4 and 4.5, it is enough to show that

$$\theta_{\mathcal{N}_M(F)\,X} = ((\rho_X \circ -) * \otimes) \circ (\mu_{X,I_{\mathcal{L}}} * \otimes) \circ \xi_{X,I_{\mathcal{L}}} \circ (\mathrm{id}_{\mathcal{N}_M(F)\,X} \times \varepsilon).$$

Recall that, by definition of $\mu_{X,I_{\mathcal{L}}}$, it holds that $(\mu_{X,I_{\mathcal{L}}} * \otimes) \circ \xi_{X,I_{\mathcal{L}}} = \gamma_{X,I_{\mathcal{L}}}$. Therefore, it now suffices to show the following equality:

$$\theta_{\mathcal{N}_M(F)\,X} = ((\rho_X \circ -) * \otimes) \circ \gamma_{X,I_{\mathcal{L}}} \circ (\mathrm{id}_{\mathcal{N}_M(F)\,X} \times \varepsilon)$$

On other words, we need to prove that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{N}_M(F)\,X \overline{\times} \mathsf{MCat}(\mathcal{C})(-,\square) \arrow[r, "\theta_{\mathcal{N}_M(F)\,X}"] \arrow[d, "(\mathrm{id}_{\mathcal{N}_M(F)\,X} \times \varepsilon)"'] & \mathcal{N}_M(F)\,X \circ \otimes \\
\mathcal{N}_M(F)\,X \overline{\times} \mathcal{N}_M(F)\,I_{\mathcal{L}} \arrow[r, "\gamma_{X,I_{\mathcal{L}}}"'] & \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} I_{\mathcal{L}}) \circ \otimes \arrow[u, "(\rho_X \circ -) * \otimes"']
\end{tikzcd} \tag{4.6}$$

Note how this diagram does not mention Day convolution anywhere, unlike Diagram 4.4 which we started with. Using the Kan extension characterisation, all Day convolutions were eliminated from the proof goal.

Finally, to show that Diagram 4.6 commutes, recall that $\mathsf{MCat}(\mathcal{C})(\mathbb{B}, []) = \emptyset$ whenever $\mathbb{B} \neq []$, so Diagram 4.6 trivially commutes when $\mathbb{B} \neq []$ is plugged into the second component. So the only interesting case is where $\mathbb{B} = []$, which is subject to the following square, where $[A_1, \dots, A_n] =: \mathbb{A} \in \mathsf{MCat}(\mathcal{C})$ and $F\,\mathbb{A} := F\,A_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,A_n$:

$$\begin{tikzcd}
\mathcal{L}(F\,\mathbb{A}, X) \times \mathsf{MCat}(\mathcal{C})([], []) \arrow[r, "\theta_{\mathcal{N}_M(F)\,X}"] \arrow[d, "(\mathrm{id}_{\mathcal{N}_M(F)\,X} \times \varepsilon)_{\mathbb{A},[]}"'] & \mathcal{L}(F\,\mathbb{A}, X) \\
\mathcal{L}(F\,\mathbb{A}, X) \times \mathcal{L}(I_{\mathcal{L}}, I_{\mathcal{L}}) \arrow[r, "(\gamma_{X, I_{\mathcal{L}}})_{\mathbb{A},[]}"'] & \mathcal{L}(F\,\mathbb{A}, X \otimes_{\mathcal{L}} I_{\mathcal{L}}) \arrow[u, "\rho_X \circ -"']
\end{tikzcd}$$

By applying the definitions of ε, γ and $\theta_{\mathcal{N}_M(F)\,X}$, one straightforwardly but tediously computes that this square commutes.

In summary, zooming out to the big picture, the left coherence diagram for $\mathcal{N}_M(F)$ holds. We still need to prove the right coherence and associativity diagrams.

The proof for the right coherence diagram is totally analogous to the proof for the left diagram.

For the associativity diagram, we again use the Kan extension characterisation of the Day convolution to eliminate all Day products from the diagram. This yields the following diagram, where $\widetilde{\otimes} : \mathsf{MCat}(\mathcal{C}) \times \mathsf{MCat}(\mathcal{C}) \times \mathsf{MCat}(\mathcal{C}) \to \mathsf{MCat}(\mathcal{C})$ is a three-fold version of $\otimes$ (since $\otimes$ is strict, the concrete definition of $\widetilde{\otimes}$ is irrelevant):

$$\begin{tikzcd}
(\mathcal{N}_M(F)\,X \mathbin{\overline{\times}} \mathcal{N}_M(F)\,Y) \mathbin{\overline{\times}} \mathcal{N}_M(F)\,C \arrow[r, "((f,g),h) \mapsto (f,(g,h))"] \arrow[d, "\gamma_{X,Y} \times \mathrm{id}_{\mathcal{N}_M(F)\,Z}"'] & \mathcal{N}_M(F)\,A \mathbin{\overline{\times}} (\mathcal{N}_M(F)\,B \mathbin{\overline{\times}} \mathcal{N}_M(F)\,C) \arrow[d, "\mathrm{id}_{\mathcal{N}_M(F)\,X} \times \gamma_{Y,Z}"] \\
(\mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} Y) \circ \otimes) \mathbin{\overline{\times}} \mathcal{N}_M(F)\,Z \arrow[d, "\gamma_{X \otimes_{\mathcal{L}} Y, Z}"'] & \mathcal{N}_M(F)\,X \mathbin{\overline{\times}} (\mathcal{N}_M(F)\,(Y \otimes_{\mathcal{L}} Z) \circ \otimes) \arrow[d, "\gamma_{X, Y \otimes_{\mathcal{L}} Z}"] \\
\mathcal{N}_M(F)\,((X \otimes_{\mathcal{L}} Y) \otimes_{\mathcal{L}} Z) \circ \widetilde{\otimes} \arrow[r, "(\alpha_{X,Y,Z} \circ -) * \widetilde{\otimes}"'] & \mathcal{N}_M(F)\,(X \otimes_{\mathcal{L}} (Y \otimes_{\mathcal{L}} Z)) \circ \widetilde{\otimes}
\end{tikzcd}$$

One can prove that this diagram commutes by faithfully applying all the definitions involved.

In summary, the coherence diagrams for $\mathcal{N}_M(F)$ are true, i.e. $\mathcal{N}_M(F)$ is lax monoidal. $\square$

Now that $\mathcal{N}_M(F)$ is lax monoidal, the glueing construction can be applied to Figure 4.1. Thanks to Lemmas 4.19 and 4.20, one obtains the following:

- The functors $U : \mathrm{id} \downarrow \mathcal{N}_M(F) \to \mathcal{L}$ and $V : \mathrm{id} \downarrow \mathcal{N}_M(F) \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ are strictly monoidal, they are even SMCC morphisms as in Definition 2.21.
- The comma category $\mathrm{id} \downarrow \mathcal{N}_M(F)$ is a symmetric monoidal closed category with tensor product $\otimes_\downarrow$ given as follows:
  - On objects, $(P, \delta, X) \otimes_\downarrow (Q, \zeta, Y) := (P \otimes_{\mathsf{Day}} Q, \mu_{X,Y} \circ (\delta \otimes_{\mathsf{Day}} \zeta), X \otimes_{\mathcal{L}} Y)$.
  - On arrows, $(\delta, f) \otimes_\downarrow (\zeta, g) := (\delta \otimes_{\mathsf{Day}} \zeta, f \otimes_{\mathcal{L}} g)$.
  - The monoidal unit is $(\mathsf{MCat}(\mathcal{C})(-, []), \varepsilon, I_{\mathcal{L}})$.

Ultimately, the aim is that in Figure 4.1, the functor $W : \mathcal{C} \to \mathrm{id} \downarrow \mathcal{N}_M(F)$ is lax monoidal. Recall that, by definition, $W C := (\mathbf{Y} C, \beta_C, F C)$ (where β as in Figure 4.1). Since the monoidal structure on $\mathrm{id} \downarrow \mathcal{N}_M(F)$ is defined componentwise, lax monoidality of W will follow straightforwardly from lax monoidality of $\mathbf{Y}$ and F. The latter functor is lax monoidal by assumption, and $\mathbf{Y}$ is lax monoidal by a similar argument to lax monoidality of $\mathcal{N}_M(F)$.

Lemma 4.25 *The functor* $\mathbf{Y} : \mathcal{C} \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ *is lax monoidal.*

Proof The proof follows exactly the same strategy as the proof that $\mathcal{N}_M(F)$ is lax monoidal.

1. One defines $\varepsilon : \mathsf{MCat}(\mathcal{C})(-, []) \Rightarrow \mathbf{Y}\, I_{\mathcal{C}}$ and $\gamma_{X,Y} : \mathbf{Y} X \overline{\times} \mathbf{Y} Y \Rightarrow \mathbf{Y}(X \otimes_{\mathcal{C}} Y) \circ \otimes_{\mathsf{MCat}(\mathcal{C})}$, where γ is natural in X, Y. Both definitions are analogous to the ones used to prove that $\mathcal{N}_M(F)$ is lax monoidal.
2. By applying the Kan extension characterisation of $\otimes_{\mathsf{Day}}$ to $\gamma_{X,Y}$, one obtains $\mu_{X,Y} : \mathbf{Y} X \otimes_{\mathsf{Day}} \mathbf{Y} Y \Rightarrow \mathbf{Y}(X \otimes_{\mathcal{C}} Y)$. While each $\mu_{X,Y}$ is a natural transformation, it is not yet clear that μ itself is natural in X, Y.
3. Using the universal property of the Day convolution, one verifies that μ is natural in X and Y, similar to Lemma 4.23.
4. Again using the universal property of the Day convolution, one verifies that ε and μ satisfy the coherence conditions by reducing the coherence diagrams to the ones only mentioning $\overline{\times}$, just as it is done in Lemma 4.24. These reduced diagrams then commute by prudently applying all the definitions. □

We are now finally in the position to show that the full functor W is lax monoidal.

Theorem 4.26 *The functor* $W : \mathcal{C} \to \mathrm{id} \downarrow \mathcal{N}_M(F)$ *is lax monoidal.*

Proof Recall that, by definition, $W X = (\mathbf{Y} X, \beta_X, F X)$, where $X \in \mathcal{C}$, and $W g = (\mathbf{Y} g, F g)$ for $g : X \to Y$ and $X, Y \in \mathcal{C}$. Further, the monoidal structure $\otimes_\downarrow$ is defined componentwise, as explained above.

The coherence morphisms for W will therefore also be defined componentwise. Let $\varepsilon^{F} : I_{\mathcal{L}} \to F\, I_{\mathcal{C}}$ and $\mu^{F}_{X,Y} : F X \otimes_{\mathcal{L}} F Y \to F(X \otimes Y)$ be the coherence morphisms for F, and $\varepsilon^{\mathbf{Y}} : \mathsf{MCat}(\mathcal{C})(-, []) \Rightarrow \mathbf{Y}\, I_{\mathcal{C}}$ as well as $\mu^{\mathbf{Y}}_{Y,X} : \mathbf{Y} X \otimes_{\mathsf{Day}} \mathbf{Y} Y \Rightarrow \mathbf{Y}(X \otimes_{\mathcal{C}} Y)$ the coherence morphisms of $\mathbf{Y}$. Also, let $\varepsilon^{\mathcal{N}_M} : \mathsf{MCat}(\mathcal{C})(-, []) \Rightarrow \mathcal{N}_M(F)\, I_{\mathcal{L}}$ and $\mu^{\mathcal{N}_M}_{Y,X} : \mathcal{N}_M(F)\, X \otimes_{\mathsf{Day}} \mathcal{N}_M(F)\, Y \Rightarrow \mathcal{N}_M(F)\,(X \otimes_{\mathcal{C}} Y)$ be the coherence morphisms of $\mathcal{N}_M(F)$.

To define $\varepsilon : I_{\mathrm{id}\downarrow\mathcal{N}_M(F)} = (\mathsf{MCat}(\mathcal{C})(-, []), \varepsilon^{\mathcal{N}_M}, I_{\mathcal{L}}) \to (\mathbf{Y}\, I_{\mathcal{C}}, \beta_{I_{\mathcal{C}}}, F\, I_{\mathcal{C}}) = W\, I_{\mathcal{C}}$, one sets $\varepsilon = (\varepsilon^{\mathbf{Y}}, \varepsilon^{F})$. To prove that this is indeed a morphism in the comma category $\mathrm{id} \downarrow \mathcal{N}_M(F)$, the following square needs to commute:

$$\begin{tikzcd}
\mathsf{MCat}(\mathcal{C})(-, []) \arrow[r, "\varepsilon^{\mathcal{N}_M}"] \arrow[d, "\varepsilon^{\mathbf{Y}}"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, I_{\mathcal{L}}) \arrow[d, "\varepsilon^{F} \circ -"] \\
\mathsf{MCat}(-, [I_{\mathcal{C}}]) \arrow[r, "\beta_{I_{\mathcal{C}}}"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, F\, I_{\mathcal{C}})
\end{tikzcd}$$

This follows by carefully applying all the definitions involved.

Next, we define $\mu_{X,Y} : W X \otimes_{\downarrow} W Y \to W(X \otimes_{\mathcal{C}} Y)$. Unfolding the definition of W shows that $\mu_{X,Y}$ needs to have type

$$(\mathbf{Y} X \otimes_{\mathsf{Day}} \mathbf{Y} Y, \mu^{\mathcal{N}_M}_{X,Y} \circ (\beta_X \otimes_{\mathsf{Day}} \beta_Y), F X \otimes_{\mathcal{L}} F Y) \to (\mathbf{Y}(X \otimes_{\mathcal{C}} Y), \beta_{X \otimes_{\mathcal{C}} Y}, F(X \otimes_{\mathcal{C}} Y)).$$

Set $\mu_{X,Y} := (\mu^{\mathbf{Y}}_{X,Y}, \mu^{F}_{X,Y})$. To prove that this is indeed a morphism in the comma category, we need to show that the following square commutes:

$$\begin{tikzcd}
\mathbf{Y} X \otimes_{\mathsf{Day}} \mathbf{Y} Y \arrow[r, "\mu^{\mathcal{N}_M}_{X,Y} \circ (\beta_X \otimes_{\mathsf{Day}} \beta_Y)"] \arrow[d, "\mu^{\mathbf{Y}}_{X,Y}"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, F X \otimes_{\mathcal{L}} F Y) \arrow[d, "\mu^{F}_{X,Y} \circ -"] \\
\mathbf{Y}(X \otimes_{\mathcal{C}} Y) \arrow[r, "\beta_{X \otimes_{\mathcal{C}} Y}"'] & \mathcal{L}(F - \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F-, F(X \otimes_{\mathcal{C}} Y))
\end{tikzcd}$$

The commutativity of the square above follows by first using the universal property of the Day convolution to replace $\otimes_{\mathsf{Day}}$ by $\overline{\times}$ (similar to how it was done in the proofs of Lemmas 4.23 and 4.24), and then applying the definitions to check that the modified diagram commutes. The verification is long and arduous, but no new insights are needed.

The next step is to show that $\mu_{X,Y}$ is natural in X, Y, i.e. that the following square commutes, where $X, Y, A, B \in \mathcal{C}$ and $f : X \to A$, $g : Y \to B$:

$$\begin{tikzcd}
W X \otimes_{\downarrow} W Y \arrow[r, "\mu_{X,Y}"] \arrow[d, "W f \otimes_{\downarrow} W g"'] & W(X \otimes_{\mathcal{C}} Y) \arrow[d, "W(f \otimes_{\mathcal{C}} g)"] \\
W A \otimes_{\downarrow} W B \arrow[r, "\mu_{A,B}"'] & W(A \otimes_{\mathcal{C}} B)
\end{tikzcd} \tag{4.7}$$

The fact that the square above commutes is induced from the naturality of both $\mu^{\mathbf{Y}}_{X,Y}$ and $\mu^{F}_{X,Y}$ in X, Y as well as the fact that the monoidal structure on $\mathrm{id} \downarrow \mathcal{N}_M(F)$ is given pointwise. Concretely, unfolding some of the definitions in Diagram 4.7 yields the following diagram:

$$\begin{tikzcd}
WX \otimes_{\downarrow} WY \arrow[r, "(\mu^{\mathbf{Y}}_{X,Y}, \mu^{F}_{X,Y})"] \arrow[d, "(\mathbf{Y} f \otimes_{\mathrm{Day}} \mathbf{Y} g, F f \otimes_{\mathcal{L}} F g)"'] & W(X \otimes_{\mathcal{C}} Y) \arrow[d, "(\mathbf{Y}(f \otimes_{\mathcal{C}} g), F(f \otimes_{\mathcal{C}} g))"] \\
WA \otimes_{\downarrow} WB \arrow[r, "(\mu^{\mathbf{Y}}_{A,B}, \mu^{F}_{A,B})"'] & W(A \otimes_{\mathcal{C}} B)
\end{tikzcd}$$

And routine calculations show that the square above commutes, using that both $\mu^{\mathbf{Y}}_{X,Y}$ and $\mu^{F}_{X,Y}$ are natural in X, Y.

The last step to prove that W is lax monoidal is to verify the coherence conditions for ε and $\mu_{X,Y}$. As in the proof that $\mu_{X,Y}$ is natural in X, Y, one reduces the coherence conditions for W to the coherence conditions for $\mathbf{Y}$ and F.

It follows that W is lax monoidal. □

It is worth summarising the results of this section. All the relevant functors and natural transformations are depicted in the following pasting diagrams:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathbf{Y}"] \arrow[d, "F"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathrm{op}}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\Leftarrow \beta"] \\
\mathcal{L} \arrow[r, "\mathcal{N}_M(F)"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathrm{op}}, \mathsf{Set}]}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[dr, dashed, "W"] \arrow[drr, bend left, "\mathbf{Y}"] \arrow[ddr, bend right, "F"'] & & \\
& (\mathrm{id} \downarrow \mathcal{N}_M(F)) \arrow[r, "V"] \arrow[d, "U"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathrm{op}}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\Leftarrow \alpha"] \\
& \mathcal{L} \arrow[r, "\mathcal{N}_M(F)"'] & {[\mathsf{MCat}(\mathcal{C})^{\mathrm{op}}, \mathsf{Set}]}
\end{tikzcd}$$

- In Section 4.5, it was shown that W is a full functor (this is Theorem 4.16).
- In Lemma 4.24, it was verified that $\mathcal{N}_M(F)$ is a lax monoidal functor.
- Using the standard fact that $[\mathsf{MCat}(\mathcal{C})^{\mathrm{op}}, \mathsf{Set}]$ has pullbacks and that $\mathcal{N}_M(F)$ is lax monoidal, Lemmas 4.19 and 4.20 yield that $\mathrm{id} \downarrow \mathcal{N}_M(F)$ admits a symmetric monoidal closed structure with tensor product $\otimes_{\downarrow}$ and that U and V are functors between SMCCs as in Definition 2.21.
- Thanks to Theorem 4.26, the functor W is lax monoidal with respect to $\otimes_{\downarrow}$.

These results give rise to the following, final theorem.

Theorem 4.27 *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathrm{I}_{\mathcal{C}})$ be a small symmetric monoidal category, $(\mathcal{L}, \otimes_{\mathcal{L}}, \mathrm{I}_{\mathcal{L}}, \multimap_{\mathcal{L}})$ an SMCC, and $F : \mathcal{C} \to \mathcal{L}$ a lax monoidal functor.*

There exists an SMCC $(\widetilde{\mathcal{L}}, \otimes_{\widetilde{\mathcal{L}}}, \mathrm{I}_{\widetilde{\mathcal{L}}}, \multimap_{\widetilde{\mathcal{L}}})$*, a full lax monoidal functor* $\widetilde{\mathsf{F}} : \mathcal{C} \to \widetilde{\mathcal{L}}$ *and functor* $\mathsf{q} : \widetilde{\mathcal{L}} \to \mathcal{L}$ *between SMCCs as in Definition 2.21 such that the following triangle commutes:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{\mathsf{F}}"] \arrow[dr, "\mathsf{F}"'] & \widetilde{\mathcal{L}} \arrow[d, "\mathsf{q}"] \\ & \mathcal{L} \end{tikzcd}$$

Proof In the notation of the paragraph above, pick $\widetilde{\mathcal{L}} := \mathrm{id} \downarrow \mathcal{N}_{\mathsf{M}}(\mathsf{F})$, set $\widetilde{\mathsf{F}} := W$, and set $\mathsf{q} := \mathsf{U}$. The required properties are then all satisfied. □

In particular, the theorem above holds when $\mathcal{C}$ has a Markov structure.

Corollary 4.28 *Let* $(\mathcal{C}, \mathsf{F}, \mathcal{L})$ *be a Banach category such that* $\mathcal{C}$ *is small.*

There exists a Banach category $(\mathcal{C}, \widetilde{\mathsf{F}}, \widetilde{\mathcal{L}})$ *and a functor* $\mathsf{q} : \widetilde{\mathcal{L}} \to \mathcal{L}$ *of SMCCs as in Definition 2.21 such that* $\widetilde{\mathsf{F}}$ *is full and the following triangle commutes:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{\mathsf{F}}"] \arrow[dr, "\mathsf{F}"'] & \widetilde{\mathcal{L}} \arrow[d, "\mathsf{q}"] \\ & \mathcal{L} \end{tikzcd}$$

The smallness condition in the corollary above seems restrictive: $\mathsf{FinStoch}$ is not small (if the collection of all finite sets was a set, say X, then $\{X\}$ is also a finite set, from which one can obtain a contradiction). Also, Stoch is not small, so one cannot fullify the examples from Section 3.3. It is future work (see Section 5.3) to prove the fullification result under weaker assumptions. We also give a brief hint on a method to still fullify $\mathsf{FinStoch}$ using Corollary 4.28 in Section 5.3, leaving the technical details as future work and noting that this method does not apply to Stoch.

It turns out that the mapping from Banach categories to full Banach categories underlying Corollary 4.28 is functorial, provided that one fixes the Markov category $\mathcal{C}$, as described below.

Definition 4.29 ($\mathsf{BanCat}(\mathcal{C})$) *Let* $\mathcal{C}$ *be a Markov category. The category* $\mathsf{BanCat}(\mathcal{C})$ *is the following subcategory of* BanCat*:*

- *Objects of* $\mathsf{BanCat}(\mathcal{C})$ *are Banach spaces whose Markov category is* $\mathcal{C}$*.*
- *A* $\mathsf{BanCat}(\mathcal{C})$*-morphism* $(\mathcal{C}, \mathsf{F}, \mathcal{L}) \to (\mathcal{C}, \mathsf{F}', \mathcal{L}')$ *is a* BanCat*-morphism of the form* $(\mathrm{id}_{\mathcal{C}}, \mathsf{G})$.

The category $\mathsf{BanCat}_{\mathsf{full}}(\mathcal{C})$ *is defined similarly.*

With this definition at hand, it is possible to show that the fullification is functorial for fixed $\mathcal{C}$.

Proposition 4.30 *Let $\mathcal{C}$ be a small Markov category. The fullification construction gives rise to a functor $\mathcal{F} : \mathsf{BanCat}(\mathcal{C}) \to \mathsf{BanCat}_{\mathsf{full}}(\mathcal{C})$.*

Proof On objects, define $\mathcal{F}(\mathcal{C}, \mathsf{F}, \mathcal{L}) := (\mathcal{C}, \widetilde{\mathsf{F}}, \widetilde{\mathcal{L}})$, following the notation of Corollary 4.28. The key part is to define the operation of $\mathcal{F}$ on arrows. This reduces to the universal property of comma categories.

Let $(\mathcal{C}, \mathsf{F}, \mathcal{L}), (\mathcal{C}, \mathsf{F}', \mathcal{L}')$ be Banach categories and $(\mathrm{id}_{\mathcal{C}}, \mathsf{G}) : (\mathcal{C}, \mathsf{F}, \mathcal{L}) \to (\mathcal{C}, \mathsf{F}', \mathcal{L}')$ a morphism in $\mathsf{BanCat}(\mathcal{C})$. That is, $\mathsf{G} : \mathcal{L} \to \mathcal{L}'$ is a functor strictly preserving the SMCC structure (see Definition 2.21) and the following triangle commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathsf{F}"] \arrow[dr, "\mathsf{F}'"'] & \mathcal{L} \arrow[d, "\mathsf{G}"] \\
& \mathcal{L}'
\end{tikzcd}$$

The goal is to define a $\mathsf{BanCat}_{\mathsf{full}}(\mathcal{C})$-morphism $\mathcal{F}(\mathrm{id}_{\mathcal{C}}, \mathsf{G}) : (\mathcal{C}, \widetilde{\mathsf{F}}, \widetilde{\mathcal{L}}) \to (\mathcal{C}, \widetilde{\mathsf{F}'}, \widetilde{\mathcal{L}'})$, which means defining a functor $\widetilde{\mathsf{G}} : \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}'}$ strictly preserving the SMCC structure such that the following triangle commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{\mathsf{F}}"] \arrow[dr, "\widetilde{\mathsf{F}'}"'] & \widetilde{\mathcal{L}} \arrow[d, "\widetilde{\mathsf{G}}"] \\
& \widetilde{\mathcal{L}'}
\end{tikzcd}$$

Recall that both $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}'}$ are comma categories. In particular, we can apply the universal property of comma categories (Proposition 4.3) to define $\widetilde{\mathsf{G}}$, which will be done in the following. To keep track of all the categories and functors involved, consider the diagram below:

$$\begin{tikzcd}
{[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Rightarrow"] & \widetilde{\mathcal{L}'} \arrow[l, "\mathsf{V}'"'] \arrow[d, "\mathsf{q}'"'] & & \widetilde{\mathcal{L}} \arrow[ll, dashed, bend right, "\exists!\widetilde{\mathsf{G}}"'] \arrow[r, "\mathsf{V}"] \arrow[d, "\mathsf{q}"] & {[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]} \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\Leftarrow"] \\
{[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]} & \mathcal{L}' \arrow[l, "\mathcal{N}_{\mathsf{M}}(\mathsf{F}')"] & \mathcal{C} \arrow[l, "\mathsf{F}'"] \arrow[r, "\mathsf{F}"'] \arrow[ul, "\widetilde{\mathsf{F}'}"'] \arrow[ur, "\widetilde{\mathsf{F}}"] & \mathcal{L} \arrow[r, "\mathcal{N}_{\mathsf{M}}(\mathsf{F})"'] \arrow[ll, bend left, "\mathsf{G}"] & {[\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]}
\end{tikzcd}$$

To apply the universal property of the comma category $\widetilde{\mathcal{L}'}$, we need to define functors $\mathsf{S} : \widetilde{\mathcal{L}} \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ and $\mathsf{T} : \widetilde{\mathcal{L}} \to \mathcal{L}'$ together with a natural transformation

$\theta : S \Rightarrow \mathcal{N}_M(F') \circ T$. We set $S := V$ (see the diagram above). Further, we define $T := G \circ q$. For the natural transformation θ, recall that objects of $\widetilde{\mathcal{L}'}$ are triples (P, ζ, L), where $P : \mathsf{MCat}(\mathcal{C}) \to \mathsf{Set}$ is a presheaf, $L \in \mathcal{L}$, and $\zeta : P \Rightarrow \mathcal{N}_M(F)\, L$ a natural transformation. The components of $\zeta_{[X_1,\ldots,X_n]}$ have type

$$P\,[X_1, \ldots, X_n] \to \mathcal{L}(F\,X_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,X_n, L)$$

To define θ, observe that $\theta_{(P,\zeta,L)}$ needs to have type

$$V\,(P, \zeta, L) = P \Rightarrow \mathcal{N}_M(F')\,(G\,L) = \mathcal{N}_M(F')\,(G(q(P, \zeta, L)))$$

Going one level down the hierarchy, $(\theta_{(P,\zeta,L)})_{[X_1,\ldots,X_n]}$ needs to have type

$$P\,[X_1, \ldots, X_n] \to \mathcal{L}'(F'\,X_1 \otimes_{\mathcal{L}'} \cdots \otimes_{\mathcal{L}'} F'\,X_n, G\,L).$$

Observing that $G \circ F = F'$, it suffices to define a function of type

$$P\,[X_1, \ldots, X_n] \to \mathcal{L}'(G\,(F\,X_1) \otimes_{\mathcal{L}'} \cdots \otimes_{\mathcal{L}'} G\,(F\,X_n), G\,L).$$

This is now straightforward: First use $\zeta_{[A_1,\ldots,A_n]}$ to get from $P\,[X_1, \ldots, X_n]$ to the hom-set $\mathcal{L}(F\,X_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,X_n, L)$, and afterwards use the operation of G on morphisms to get from $\mathcal{L}(F\,X_1 \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} F\,X_n, L)$ to $\mathcal{L}'(G\,(F\,X_1) \otimes_{\mathcal{L}'} \cdots \otimes_{\mathcal{L}'} G\,(F\,X_n), G\,L)$, observing that G strictly preserves tensor products, i.e. $G\,A \otimes_{\mathcal{L}'} G\,B = G\,(A \otimes_{\mathcal{L}} B)$ for all A, B.

It is routine to verify that θ is a well-defined natural transformation $V \Rightarrow \mathcal{N}_M(F') \circ G \circ q$. From the universal property of the comma category, we obtain the desired functor $\widetilde{G}$. The fact that $\widetilde{G}$ strictly preserves the SMCC structure follows from its definition (see paragraph below Proposition 4.3) and the fact that both V (by Lemma 4.20) as well as $T \circ q$ (as composition) strictly preserve the SMCC structure.

The fact that $\mathcal{F}$ is functorial, i.e. that it respects identities and compositions, follows by prudently applying all the definitions involved. □

Chapter 5

Conclusion

In this thesis, we investigated and gave a name to Banach categories, motivated by the fact that they can be used to nicely give semantics to probabilistic programming languages that combine higher-order functionality and non-linearity via a modality [15]. This is usually considered a highly technical and subtle task, as witnessed by Ehrhard and colleagues [12, 21, 18] as well as Dahlqvist and Kozen [10].

Using the well-known calculus of string diagrams [53, 7, 50], which gives a convenient method to reason about morphisms in monoidal categories, we analysed how in a Banach category $(\mathcal{C}, F, \mathcal{L})$, the Markov structure on $\mathcal{C}$ interacts with the lax monoidal structure of F. We found out that strong monoidal functors have even better graphical properties (Section 3.2). To motivate this diagrammatic analysis, we gave a graphical description of convex spaces [59, 26], highlighting that the diagrams provide a concise and easy to parse method of reasoning (Section 3.1).

Motivated by the fact that the semantics of Azevedo de Amorim's programming language [15] behaves better when the lax monoidal functor is full, we proved that Banach categories can be fullified in the following sense: For any Banach category $(\mathcal{C}, F, \mathcal{L})$ such that $\mathcal{C}$ is small, there exists another Banach category $(\mathcal{C}, \widetilde{F}, \widetilde{\mathcal{L}})$ and a functor $q : \mathcal{L} \to \widetilde{\mathcal{L}}$ such that $\widetilde{F}$ is full and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & \widetilde{\mathcal{L}} \arrow[d, "q"] \\
& \mathcal{L}
\end{tikzcd}$$

This is subject to Corollary 4.28 and the main contribution of the thesis. Lots of time and effort was spent on understanding and dealing with the Day convolution. We even proved that this construction is functorial. To only obtain the functors $\widetilde{F}$ and q as well as the category $\widetilde{\mathcal{L}}$, but not any monoidality results, the assumption that $\mathcal{C}$ is small is not needed (Corollary 4.17).

5.1 Note on the use of $\mathsf{MCat}(\mathcal{C})$

The fullification of a Banach category $(\mathcal{C}, \mathsf{F}, \mathcal{L})$ relied on the use of $\mathsf{MCat}(\mathcal{C})$ (Definition 4.4), which is relatively intricate. In particular, the multicategorical nerve $\mathcal{N}_{\mathsf{M}}(\mathsf{F}) : \mathcal{L} \to [\mathsf{MCat}(\mathcal{C})^{\mathsf{op}}, \mathsf{Set}]$ (Definition 4.13) turned out to be difficult to handle in the proofs as a result.

To only obtain the full functor, i.e. the result claimed in Corollary 4.17, these technicalities can be avoided by replacing all occurrences of $\mathsf{MCat}(\mathcal{C})$ by $\mathcal{C}$. Instead of the multicategorical nerve, one can use the ordinary nerve of F. This is a functor $\mathcal{N}(\mathsf{F}) : \mathcal{L} \to [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ defined as $\mathcal{N}(\mathsf{F})\,\mathsf{L} := \mathcal{L}(\mathsf{F}-, \mathsf{L})$ on objects. However, one cannot prove that $\mathcal{N}(\mathsf{F})$ is lax monoidal with respect to the Day convolution, making it impossible to show the main theorem of the present thesis (Corollary 4.28). With the generalisation obtained by using $\mathsf{MCat}(\mathcal{C})$, however, the result becomes provable.

5.2 Related Work

Fullification of a Cartesian functor. There is work on fullifying Cartesian functors from Cartesian categories to closed bi-Cartesian categories. Roughly speaking, this setting is a special case of a Banach category where the monoidal products are given by actual products in the sense of Definition 2.3. The mentioned related work arises in the scope of the **λ-definability problem**, which concerns the definiability of semantic elements in the simply-typed λ-calculus [6]. Alimohamed [1] was first to study the setting outlined above, working in λ-calculus with only function types, extending previous work by Jung and Tiuryn [37] who approach the same problem, but only account for Henkin models, not Cartesian categories in general. Katsumata [39] later extended Alimohamed's construction to also allow for sum-types, a subtly more difficult problem. The rough setup of both fullification constructions is similar, and Katsumata additionally gives a comprehensive overview the construction, which is summaried in Section 4.1.

Notably different to the fullification construction presented in this thesis is that Alimohamed and Katsumata use a pullback instead of a comma category. We believe that it is possible to use pullbacks to also fullify Banach categories. We have, however, opted for comma categories as we conjecture that the fullification construction given in the present thesis admits a universal property, see future work (Section 5.3).

Languages for Banach categories. Azevedo de Amorim [15] introduces a probabilistic programming language that can be interpreted by Banach categories. Azevedo de Amorim does not use the term "Banach category", but instead refers to tuples of Markov categories, SMCCs and lax monoidal functors between them.

Azevedo de Amorim's language consists of two levels: One level is a higher-order linear language interpreted by an SMCC, and the other level is a Markov kernel language interpreted by a Markov category. The lax monoidal functor is used to mediate between both levels. The language can be understood as follows: Variables of ground type in the linear level range over probability distributions, and can hence be used only once. On the other hand, variables of ground type in the Markov kernel language range over values that were already sampled from a distribution, which can therefore be copied freely. Translation from the linear level to the Markov level can be seen as sampling from a distribution. One uses a distribution from the linear level, samples from it, and the resulting sample can then be used in the Markov language.

Further, Azevedo de Amorim studies three models of his language. The first is a Banach category which can be seen as a generalisation of the Banach category $\mathsf{FinStoch} \to \mathsf{FinDimVect}_{\mathbb{R}}$ given in Section 3.3.1 to the countably infinite case, using so-called probabilistic coherence spaces [12]. The second one is $\mathsf{Stoch} \to \mathsf{RoBan}$ which we outlined in Section 3.3.2. The last example uses Eilenberg-Moore algebras [25] over monoidal monads as SMCC and corresponding Kleisli categories [40] as Markov category. The latter construction is motivated by the fact that Markov categories often arise from Kleisli categories, as noted by Fritz [27, Section 3]. For example, Stoch arises as Kleisli category of the Giry monad [29] (see e.g. Cho and Jacobs [5, Example 2.5]).

Banach spaces and probabilistic programming languages. The key motivation for the term *Banach category* is drawn from the fact that the SMCC in a Banach category is often some kind of category of Banach spaces, i.e. normed vector spaces that are closed. This goes far back to the early years of probabilistic programming languages, where Kozen [41] gave semantics to an imperative probabilistic first-order language using Banach spaces of measures. More recently, Dahlqvist and Kozen [10] give an extensive treatment of semantics for a higher-order linear languages whose models are given as certain categories of Banach spaces. Ultimately, they define the category RoBan (see Definition 3.11), which turned out to be a versatile model for higher-order linear languages, also used as part of a model to Azevedo de Amorim's [15] language. In particular, RoBan is appealing since ordered Banach space have a rich theory having been studied for multiple decades.

Dahlqvist and Kozen also consider *Banach lattices*, which form a non-closed subcategory of RoBan. Banach lattices have long been used to give semantics to probabilistic languages that are not higher order, for instance by Kozen in 1981 [41]. Recently, Azevedo de Amorim, Witzman, and Kozen [16] observed that a subcategory of Banach lattices, namely perfect Banach lattices with positive linear functions of norm at most one, nicely generalises probabilistic coherence spaces to the continuous case, and can be used as the SMCC of a Banach category.

5.3 Future Directions

Fullification as adjunction. We have seen in Proposition 4.30 that the fullification is a functor $\mathcal{F} : \mathsf{BanCat}(\mathcal{C}) \to \mathsf{BanCat}_{\mathsf{full}}(\mathcal{C})$. There is also the obvious forgetful functor $U : \mathsf{BanCat}_{\mathsf{full}}(\mathcal{C}) \to \mathsf{BanCat}(\mathcal{C})$. We conjecture that $\mathcal{F}$ is right adjoint to U. This would give rise to bijections between the hom-sets $\mathsf{BanCat}(\mathcal{C})(U A, B) \cong \mathsf{BanCat}_{\mathsf{full}}(\mathcal{C})(A, \mathcal{F} B)$, natural in A and B. This is motivated by the fact that in many settings there are adjunctions involving forgetful functors, for instance between free functors and forgetful functors.

Smallness of the Markov category. Our main result (Corollary 4.28) relies on the Markov category being small. This severely limits the applicability of Corollary 4.28, and even the Banach category with $\mathsf{FinStoch}$ as Markov category cannot be used since $\mathsf{FinStoch}$ is not small: The objects of $\mathsf{FinStoch}$ are finite sets, and it is clear that the set of all finite sets is not a set (if it was as set, say X, then $\{X\}$ is also a finite set which easily leads to a contradiction). One can study the full subcategory $\mathsf{FinStoch}_{\mathbb{N}}$ of $\mathsf{FinStoch}$ where the objects are sets of the form $\{1, \ldots, n\}$ for $n \in \mathbb{N}$. It is clear that $\mathsf{FinStoch}_{\mathbb{N}}$ is small, and one does not lose any generality by working with $\mathsf{FinStoch}_{\mathbb{N}}$ instead of $\mathsf{FinStoch}$ (in fact, $\mathsf{FinStoch}$ and $\mathsf{FinStoch}_{\mathbb{N}}$ are equivalent[1]). However, when moving on to the more interesting case of continuous probability, i.e. Stoch, then there is no way of "smallifying" the category without losing a significant amount of generality. In the future, it would be interesting to analyse whether there exist weaker sufficient conditions for the fullification. This would reduce to analysing sufficient conditions for the coend in the Day convolution (Definition 4.7).

Kan extension and Day convolution. Speaking of Day convolution ($\otimes_{\mathsf{Day}}$), it turned out that it was rather difficult to work with the Kan extension characterisation (Proposition 4.10). On the one hand, this characterisation is extremely convenient as it can be phrased in basic categorical terms, unlike the conend formulation. On the other hand, however, the operation of $\otimes_{\mathsf{Day}}$ on arrows, or the monoidal structure isos of $\otimes_{\mathsf{Day}}$, are not stated with respect to the Kan extension characterisation in the literature (to the best of our knowledge). It would be a small, yet helpful, contribution to the field to formally phrase the opereation of $\otimes_{\mathsf{Day}}$ on arrows and its monoidal structure isos with respect to the Kan extension characterisation. It will certainly help newcomers to understand the Day convolution more easily.

2-categorical perspective. Also, many of the concepts touched in this thesis are examples of 2-categories [49, 4, 17], where one has a second level of morphism. That is, 2-categories have objects, morphisms between objects (1-morphisms), and a second level of 2-morphisms between 1-morphisms. Think of categories, functors, and natural transformations. See e.g. Johnson and Yau [36] for a contemporary

[1] There is a formal notion of equivalence of categories, see e.g. Borceux [2].

overview. One could study the results of this thesis from a 2-categorical perspective and investigate how the results fit within the scope of 2-category theory.

Convex Spaces. In many examples of Banach categories, the SMCC is some kind of category of Banach spaces, so in particular a category of normed vector spaces. As mentioned in the section on convex spaces (Section 3.1), normed vector spaces are examples of convex spaces. One may wish to understand whether there are any interesting additional properties if one considers Banach categories where the SMCC additionally admits a convex structure.

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