Systems for Propositional Logics

Hai Dang

08 October, 2015

Abstract

This report lists the various systems for both classical and intuitionistic propositional logics. These include the natural deduction systems, Gentzen's sequent calculi, Hilbert's axiomatic systems, and Fitting's tableau-based systems. Fundamental results for these systems include cut-elimination, their equivalences and their decidability. We work with lists, instead of multisets, of formulas to formalize these results in Coq.

1 Basic definitions

Definition 1 (Propositional formulas)**.** The grammar of propositional formulas is

$$s, t ::= x | s \to t | s \land t | s \lor t | \bot$$

where x ranges over propositional variables. Define $\neg s = s \to \bot$ and $s \leftrightarrow t = s \to t \land t \to s$. Γ denotes a finite list of formulas.

Definition 2 (Entailment relation)**.** An entailment relation is a relation between two sets of formulas. In the natural deduction systems, the intuitionistic Gentzen system, and the Hilbert systems, it is a relation between a set of formulas (possibly empty) on the left-hand side with exactly one formula on the right-hand side. In the classical Gentzen system and Fitting's tableau system, it is a relation between two sets of an arbitrary number of formulas.

We often write $\vdash s$ to mean that the left-hand side is empty: $\emptyset \vdash s$. When both sides are empty, we will write explicitly as $\emptyset \vdash \emptyset$.

Definition 3 (Substitution)**.** A substitution is a mapping σ from variables to formulas. A substitution operation can be defined recursively for formulas. We write σs for the substitution of σ on s, and $\sigma\Gamma$ for a list of formulas of which each is a substitution of σ on each formula in Γ.

Definition 4 (Substitutability)**.** An entailment relation $\vdash$ satisfies substitutability if $\sigma\Gamma \vdash \sigma s$ if $\Gamma \vdash s$, or $\sigma\Gamma \vdash \sigma\Delta$ if $\Gamma \vdash \Delta$.

$$\frac{s \in \Gamma}{\Gamma \vdash^c s}\ \mathrm{A} \qquad \frac{\neg s, \Gamma \vdash^c \bot}{\Gamma \vdash^c s}\ \mathrm{C}$$

$$\frac{s, \Gamma \vdash^c t}{\Gamma \vdash^c s \to t}\ {\to}\mathrm{I} \qquad \frac{\Gamma \vdash^c s \to t \qquad \Gamma \vdash^c s}{\Gamma \vdash^c t}\ {\to}\mathrm{E}$$

$$\frac{\Gamma \vdash^c s \qquad \Gamma \vdash^c t}{\Gamma \vdash^c s \wedge t}\ {\wedge}\mathrm{I} \qquad \frac{\Gamma \vdash^c s \wedge t \qquad s, t, \Gamma \vdash^c u}{\Gamma \vdash^c u}\ {\wedge}\mathrm{E}$$

$$\frac{\Gamma \vdash^c s}{\Gamma \vdash^c s \vee t}\ {\vee}\mathrm{IL} \qquad \frac{\Gamma \vdash^c t}{\Gamma \vdash^c s \vee t}\ {\vee}\mathrm{IR}$$

$$\frac{\Gamma \vdash^c s \vee t \qquad s, \Gamma \vdash^c u \qquad t, \Gamma \vdash^c u}{\Gamma \vdash^c u}\ {\vee}\mathrm{E}$$

Figure 1: Rules for the classical natural deduction system **Nc**

2 Natural deduction systems

Definition 5 (Classical natural deduction system **Nc**)**.** The classical natural deduction system defines the entailment relation $\Gamma \vdash^c s$ by rules given in Figure 1.

Definition 6 (Intuitionistic natural deduction system **Ni**)**.** The intuitionistic natural deduction system defines the entailment relation $\Gamma \vdash^i s$ by rules given in Figure 2.

$$\frac{s \in \Gamma}{\Gamma \vdash^i s}\ \mathrm{A} \qquad \frac{\Gamma \vdash^i \bot}{\Gamma \vdash^i s}\ \mathrm{E}$$

$$\frac{s, \Gamma \vdash^i t}{\Gamma \vdash^i s \to t}\ {\to}\mathrm{I} \qquad \frac{\Gamma \vdash^i s \to t \qquad \Gamma \vdash^i s}{\Gamma \vdash^i t}\ {\to}\mathrm{E}$$

$$\frac{\Gamma \vdash^i s \qquad \Gamma \vdash^i t}{\Gamma \vdash^i s \wedge t}\ {\wedge}\mathrm{I} \qquad \frac{\Gamma \vdash^i s \wedge t \qquad s, t, \Gamma \vdash^i u}{\Gamma \vdash^i u}\ {\wedge}\mathrm{E}$$

$$\frac{\Gamma \vdash^i s}{\Gamma \vdash^i s \vee t}\ {\vee}\mathrm{IL} \qquad \frac{\Gamma \vdash^i t}{\Gamma \vdash^i s \vee t}\ {\vee}\mathrm{IR}$$

$$\frac{\Gamma \vdash^i s \vee t \qquad s, \Gamma \vdash^i u \qquad t, \Gamma \vdash^i u}{\Gamma \vdash^i u}\ {\vee}\mathrm{E}$$

Figure 2: Rules for the intuitionistic natural deduction system **Ni**

Fact 7 (Reflexivity of **N**[**ci**])**.** If $s \in \Gamma$ then $\Gamma \vdash s$.

Fact 8 (Substitutability of **N**[**ci**])**.** **N**[**ci**] satisfy substitutability.

Lemma 9. *Weakening and Cut are admissible in* **N**[**ci**].

$$\frac{\Gamma \vdash s \qquad \Gamma \subseteq \Gamma'}{\Gamma' \vdash s}\ \mathrm{Weak} \qquad \frac{\Gamma \vdash s \qquad s, \Gamma \vdash t}{\Gamma \vdash t}\ \mathrm{Cut}$$

Proof. By induction on the derivation $\Gamma \vdash s$. ☐

Lemma 10. *If* $\Gamma \vdash^i s$ *then* $\Gamma \vdash^c s$.

Proof. By induction on the derivation $\Gamma \vdash^i s$. ☐

Lemma 11 (Glivenko)**.** $\Gamma \vdash^c s$ *iff* $\Gamma \vdash^i \neg\neg s$.

Proof. By induction on the derivation $\Gamma \vdash^c s$. ☐

Collorary 12 (Refutation equivalence)**.** $\Gamma \vdash^c \bot$ iff $\Gamma \vdash^i \bot$.

Collorary 13 (Embedding of **Nc** in **Ni**)**.** $\Gamma \vdash^c s$ iff $\neg s, \Gamma \vdash^i \bot$.

Theorem 14. *Classical natural deduction entailment is decidable.*

3 Gentzen's sequent calculi

Definition 15 (Classical sequent system **Gc**)**.** The classical sequent system defines the entailment relation $\Gamma \Rightarrow^c \Delta$ by rules given in Figure 3. The system allows arbitrary number of formulas in the right-hand sided sequent.

$$\frac{x \in \Gamma \qquad x \in \Delta}{\Gamma \Rightarrow^c \Delta}\,\text{A} \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow^c \Delta}\,\text{E}$$

$$\frac{s \to t \in \Delta \qquad s, \Gamma \Rightarrow^c t, \Delta}{\Gamma \Rightarrow^c \Delta}\,\to\text{R} \qquad \frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow^c s, \Delta \qquad t, \Gamma \Rightarrow^c \Delta}{\Gamma \Rightarrow^c \Delta}\,\to\text{L}$$

$$\frac{s \wedge t \in \Delta \qquad \Gamma \Rightarrow^c s, \Delta \qquad \Gamma \Rightarrow^c t, \Delta}{\Gamma \Rightarrow^c \Delta}\,\wedge\text{R} \qquad \frac{s \wedge t \in \Gamma \qquad s, t, \Gamma \Rightarrow^c \Delta}{\Gamma \Rightarrow^c \Delta}\,\wedge\text{L}$$

$$\frac{s \vee t \in \Delta \qquad \Gamma \Rightarrow^c s, t, \Delta}{\Gamma \Rightarrow^c \Delta}\,\vee\text{R} \qquad \frac{s \vee t \in \Gamma \qquad s, \Gamma \Rightarrow^c \Delta \qquad t, \Gamma \Rightarrow^c \Delta}{\Gamma \Rightarrow^c \Delta}\,\vee\text{L}$$

Figure 3: Rules for the classical Gentzen system **Gc**

Definition 16 (Intuitionistic sequent system **Gi**)**.** The intuitionistic sequent system defines the entailment relation $\Gamma \Rightarrow^i s$ by rules given in Figure 4.

Fact 17 (Reflexivity of **Gi**)**.** If $s \in \Gamma$ then $\Gamma \Rightarrow^i s$.

Fact 18 (Double negation rejected in **Gi**)**.** $\Gamma \not\Rightarrow^i \neg\neg x \to x$.

Lemma 19. *Weakening is admissible in* **G**[**ci**].

$$\frac{\Gamma \Rightarrow^c \Delta \qquad \Gamma \subseteq \Gamma' \qquad \Delta \subseteq \Delta'}{\Gamma' \Rightarrow^c \Delta'}\,c\text{Weak} \qquad \frac{\Gamma \Rightarrow^i s \qquad \Gamma \subseteq \Gamma'}{\Gamma' \Rightarrow^i s}\,i\text{Weak}$$

$$\frac{s \in \Gamma}{\Gamma \Rightarrow^i s}\ \text{A} \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow^i s}\ \text{E}$$

$$\frac{s, \Gamma \Rightarrow^i t}{\Gamma \Rightarrow^i s \to t}\ {\to}\text{R} \qquad \frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow^i s \qquad t, \Gamma \Rightarrow^i u}{\Gamma \Rightarrow^i u}\ {\to}\text{L}$$

$$\frac{\Gamma \Rightarrow^i s \qquad \Gamma \Rightarrow^i t}{\Gamma \Rightarrow^i s \wedge t}\ \wedge\text{R} \qquad \frac{s \wedge t \in \Gamma \qquad s, t, \Gamma \Rightarrow^i u}{\Gamma \Rightarrow^i u}\ \wedge\text{L}$$

$$\frac{\Gamma \Rightarrow^i s}{\Gamma \Rightarrow^i s \vee t}\ \vee\text{R1} \qquad \frac{\Gamma \Rightarrow^i t}{\Gamma \Rightarrow^i s \vee t}\ \vee\text{R2}$$

$$\frac{s \vee t \in \Gamma \qquad s, \Gamma \Rightarrow^i u \qquad t, \Gamma \Rightarrow^i u}{\Gamma \Rightarrow^i u}\ \vee\text{L}$$

Figure 4: Rules for the intuitionistic Gentzen system **Gi**

Proof. By induction on the derivation of the first premise. □

Lemma 20 (Consistency of **Gc**). *$\emptyset \not\Rightarrow^c \emptyset$, and $\not\Rightarrow^c \bot$, and $\not\Rightarrow^c x$.*

Lemma 21 (Consistency of **Gi**). *$\not\Rightarrow^i \bot$ and $\not\Rightarrow^i x$.*

Lemma 22 (Generalized cut for **Gc**). *The generalized cut rule is admissible in* **Gc**.

$$\frac{\Gamma \Rightarrow^c \Delta \qquad \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \setminus s \Rightarrow^c \Delta \setminus s, \Delta'}\ \text{GCut}$$

Proof. The proof needs laboring with a lot of technical details. We need 3 nested structural inductions: first and induction on the cut formula s, then induction on $\Gamma \Rightarrow^c \Delta$, and if needed, another nested induction on $\Gamma' \Rightarrow^c \Delta'$.

The base cases (variables and Falsehood) are straightforward and do not need the 3rd induction. For implication, conjunction, and disjunction, after the 2nd induction, most cases are also straightforward. The cases below are not so, and need the 3rd induction. In all of these cases, which are for $s \to t$, $s \wedge t$ and $s \vee t$, we have 2 inductive hypotheses from the 1st induction:

$$\forall\ \Gamma\ \Delta\ \Gamma'\ \Delta'.\ \ \Gamma \Rightarrow^c \Delta \quad \longrightarrow \quad \Gamma' \Rightarrow^c \Delta' \quad \longrightarrow \quad \Gamma, \Gamma' \setminus s \Rightarrow^c \Delta \setminus s, \Delta' \tag{22.1}$$

$$\forall\ \Gamma\ \Delta\ \Gamma'\ \Delta'.\ \ \Gamma \Rightarrow^c \Delta \quad \longrightarrow \quad \Gamma' \Rightarrow^c \Delta' \quad \longrightarrow \quad \Gamma, \Gamma' \setminus t \Rightarrow^c \Delta \setminus t, \Delta' \tag{22.2}$$

From the 2nd induction, we have the following important cases for the cut formula:

(i) $s \to t$

We have $\Gamma' \Rightarrow^c \Delta'$ and some s', t' such that $s' \to t' \in \Delta$, and $s', \Gamma \Rightarrow^c t', \Delta$. The inductive hypothesis from the 2nd induction is

$$s', \Gamma, \Gamma' \setminus (s \to t) \Rightarrow^c ((t', \Delta) \setminus s \to t), \Delta' \tag{22.3}$$

We need to show $\Gamma, \Gamma' \setminus s \to t \Rightarrow^c \Delta \setminus s \to t, \Delta'$.

If $s \to t \neq s' \to t'$, then $s' \to t' \in \Delta \setminus s \to t, \Delta'$. By $\to$R we need to show $s', \Gamma, \Gamma' \setminus s \to t \Rightarrow^c t', \Delta \setminus s \to t, \Delta'$, which follows from Weakening of 22.3.

Now, consider that $s \to t = s' \to t'$, i.e. $s = s'$ and $t = t'$. We need the 3rd induction on $\Gamma' \Rightarrow^c \Delta'$. The base subcases are straightforward. In the other 7 subcases we all have $s \to t \in \Delta$, and $s, \Gamma \Rightarrow^c t, \Delta$. 22.3 becomes:

$$s, \Gamma, \Gamma' \setminus (s \to t) \Rightarrow^c ((t, \Delta) \setminus s \to t), \Delta' \tag{22.4}$$

The 7 subcases are:

(a) $s_1 \to s_2 \in \Gamma'$. We also have $\Gamma' \Rightarrow^c s_1, \Delta'$ and $s_2, \Gamma' \Rightarrow^c \Delta'$. The 2 new inductive hypotheses are:

$$\Gamma, \Gamma' \setminus s \to t \Rightarrow^c (\Delta \setminus s \to t), s_1, \Delta' \tag{22.5}$$

$$\Gamma, (s_2, \Gamma') \setminus s \to t \Rightarrow^c (\Delta \setminus s \to t), \Delta' \tag{22.6}$$

If $s \to t \neq s_1 \to s_2$, then $s_1 \to s_2 \in (\Gamma, \Gamma' \setminus s \to t)$. By $\to$L, we only have to show $\Gamma, \Gamma' \setminus s \to t \Rightarrow^c s_1, (\Delta \setminus s \to t), \Delta'$ and $s_2, \Gamma, \Gamma' \setminus s \to t \Rightarrow^c \Delta \setminus s \to t, \Delta'$, which follow from Weakening of 22.5 and 22.6, respectively.

Now, consider the case where $s \to t = s_1 \to s_2$, i.e. $s = s_1$ and $t = s_2$. Let

$$\begin{aligned}
\Gamma_1 &:= \Gamma, \Gamma' \setminus s \to t \\
\Delta_1 &:= (\Delta \setminus s \to t), \Delta' \\
\Gamma_2 &:= \Gamma, (t, \Gamma') \setminus s \to t \\
\Delta_2 &:= (\Delta \setminus s \to t), s, \Delta' \\
\Gamma_3 &:= s, \Gamma, \Gamma' \setminus s \to t \\
\Delta_3 &:= ((t, \Delta) \setminus s \to t), \Delta'
\end{aligned}$$

From 22.5 and 22.6, using 22.1, we have

$$\Gamma_1, \Gamma_2 \setminus s \Rightarrow^c (\Delta_2 \setminus s), \Delta_1 \tag{22.7}$$

Also from 22.5 and 22.4, with 22.1:

$$\Gamma_1, \Gamma_3 \setminus s \Rightarrow^c (\Delta_2 \setminus s), \Delta_3 \tag{22.8}$$

Now from 22.8 and 22.7, using 22.2, we finally have

$$\Gamma_1, (\Gamma_3 \setminus s), (\Gamma_1, \Gamma_2 \setminus s) \setminus t \Rightarrow^c (((\Delta_2 \setminus s), \Delta_3) \setminus t), (\Delta_2 \setminus s), \Delta_1 \tag{22.9}$$

whose Weakening is what we need to prove: $\Gamma_1 \Rightarrow^c \Delta_1$.

(b) $s_1 \to s_2 \in \Delta'$. We also have the inductive hypothesis

$$\Gamma, (s_1, \Gamma') \setminus s \to t \Rightarrow^c (\Delta \setminus s \to t), s_2, \Delta' \tag{22.10}$$

Since $s_1 \to s_2 \in (\Delta \setminus s \to t), \Delta'$, by $\to$R we only need to show $s_1, \Gamma, \Gamma' \setminus s \to t \Rightarrow^c s_2, (\Delta \setminus s \to t), \Delta'$, which is the Weakening of the inductive hypothesis 22.10.

(c) $s_1 \wedge s_2 \in \Gamma'$. We also have the inductive hypothesis

$$\Gamma, (s_1, s_2, \Gamma') \setminus s \to t \Rightarrow^c (\Delta \setminus s \to t), \Delta' \tag{22.11}$$

Since $s_1 \wedge s_2 \in (\Delta \setminus s \to t), \Delta'$, by $\wedge$L we only need to show $s_1, s_2, \Gamma, \Gamma' \setminus s \to t \Rightarrow^c (\Delta \setminus s \to t), \Delta'$, which is the Weakening of the inductive hypothesis 22.11.

(d) $s_1 \wedge s_2 \in \Delta'$. Similar to the case (ic).

(e) $s_1 \vee s_2 \in \Gamma'$. Similar to the case (ic).

(f) $s_1 \vee s_2 \in \Delta'$. Similar to the case (ic).

(g) $\Gamma' \equiv \Gamma_0$ and $\Delta' \equiv \Delta_0$. We have the inductive hypothesis

$$\Gamma, \Gamma' \setminus s \to t \Rightarrow^c (\Gamma \setminus s \to t), \Delta' \tag{22.12}$$

whose Weakening is what we need to prove: $\Gamma, \Gamma_0 \setminus s \to t \Rightarrow^c (\Gamma \setminus s \to t), \Delta_0$.

(ii) $s \wedge t$

In the same fashion as in case (i), in this case the most important subcase of the 3rd induction is $s_1 \wedge s_2 \in \Gamma'$. We have the followings: $s \wedge t \in \Delta$, and $\Gamma \Rightarrow^c s, \Delta$, and $\Gamma \Rightarrow^c t, \Delta$, and 2 inductive hypotheses from the 2nd induction:

$$\Gamma, \Gamma' \setminus s \wedge t \Rightarrow^c (s, \Delta) \setminus s \wedge t, \Delta' \tag{22.13}$$

$$\Gamma, \Gamma' \setminus s \wedge t \Rightarrow^c (t, \Delta) \setminus s \wedge t, \Delta' \tag{22.14}$$

For the subcase we also have $s_1 \wedge s_2 \in \Gamma'$, and $s_1, s_2, \Gamma' \Rightarrow^c \Delta'$, and the inductive hypothesis

$$\Gamma, ((s_1, s_2, \Gamma') \setminus s \wedge t) \Rightarrow^c (\Delta \setminus s \wedge t), \Delta' \tag{22.15}$$

We have to show $\Gamma, (\Gamma' \setminus s \wedge t) \Rightarrow^c (\Delta \setminus s \wedge t), \Delta'$.

If $s \wedge t \neq s_1 \wedge s_2$, then $s_1 \wedge s_2 \in (\Gamma, \Gamma' \setminus s \wedge t)$. By $\wedge$L, we only have to show $s_1, s_2, \Gamma, (\Gamma' \setminus s \wedge t) \Rightarrow^c (\Delta \setminus s \wedge t), \Delta'$, which is the Weakening of the inductive hypothesis 22.15.

If $s \wedge t = s_1 \wedge s_2$, then $s = s_1$ and $t = s_2$.

Let

$$\begin{aligned} \Gamma_1 &:= \Gamma, \Gamma' \setminus s \wedge t \\ \Delta_1 &:= (\Delta \setminus s \wedge t), \Delta' \\ \Gamma_2 &:= \Gamma, (s, t, \Gamma') \setminus s \wedge t \\ \Delta_s &:= ((s, \Delta) \setminus s \wedge t), \Delta' \\ \Delta_t &:= ((t, \Delta) \setminus s \wedge t), \Delta' \end{aligned}$$

Now, from 22.14 and 22.15, using 22.2, we have

$$\Gamma_1, \Gamma_2 \setminus t \Rightarrow^c (\Delta_t \setminus t), \Delta_1 \tag{22.16}$$

With 22.13 and 22.16, using 22.1, we finally have

$$\Gamma_1, (\Gamma_1, \Gamma_2 \setminus t) \setminus s \Rightarrow^c (\Delta_s \setminus s), (\Delta_t \setminus t), \Delta_1 \tag{22.17}$$

of which Weakening is what we need to show.

(iii) $s \vee t$

Similarly, in this case the most important subcase of the 3rd induction is $s_1 \vee s_2 \in \Gamma'$. We have the followings: $s \vee t \in \Delta$, and $\Gamma \Rightarrow^c s, t, \Delta$, and the inductive hypothesis from the 2nd induction:

$$\Gamma, \Gamma' \setminus s \vee t \Rightarrow^c (s, t, \Delta) \setminus s \vee t, \Delta' \tag{22.18}$$

For the subcase we also have $s_1 \vee s_2 \in \Gamma'$, and $s_1, \Gamma' \Rightarrow^s \Delta'$, and $s_2, \Gamma' \Rightarrow^s \Delta'$, and 2 inductive hypotheses

$$\Gamma, ((s_1, \Gamma') \setminus s \vee t) \Rightarrow^c (\Delta \setminus s \vee t), \Delta' \tag{22.19}$$

$$\Gamma, ((s_2, \Gamma') \setminus s \vee t) \Rightarrow^c (\Delta \setminus s \vee t), \Delta' \tag{22.20}$$

We have to show $\Gamma, (\Gamma' \setminus s \vee t) \Rightarrow^c (\Delta \setminus s \vee t), \Delta'$.

If $s \vee t \neq s_1 \vee s_2$, then $s_1 \vee s_2 \in (\Gamma, \Gamma' \setminus s \vee t)$. By $\vee$L, we only have to show $s_1, \Gamma, (\Gamma' \setminus s \wedge t) \Rightarrow^c (\Delta \setminus s \wedge t), \Delta'$, and $s_2, \Gamma, (\Gamma' \setminus s \wedge t) \Rightarrow^c (\Delta \setminus s \wedge t), \Delta'$, which are the Weakening of the inductive hypotheses 22.19 and 22.20, respectively.

If $s \vee t = s_1 \vee s_2$, then $s = s_1$ and $t = s_2$.

Let

$$\begin{aligned}
\Gamma_1 &:= \Gamma, \Gamma' \setminus s \vee t \\
\Delta_1 &:= (\Delta \setminus s \vee t), \Delta' \\
\Delta_{s,t} &:= ((s, t, \Delta) \setminus s \vee t), \Delta' \\
\Gamma_s &:= \Gamma, (s, \Gamma') \setminus s \vee t) \\
\Gamma_t &:= \Gamma, (t, \Gamma') \setminus s \vee t)
\end{aligned}$$

From 22.18 and 22.20, using 22.2 we have

$$\Gamma_1, \Gamma_t \setminus t \Rightarrow^c (\Delta_{s,t} \setminus t), \Delta_1 \tag{22.21}$$

Then from 22.21 and 22.19, using 22.1 we finally have

$$\Gamma_1, (\Gamma_t \setminus t), (\Gamma_s \setminus, s) \Rightarrow^c (((\Delta_{s,t} \setminus t), \Delta_1) \setminus s), \Delta_1 \tag{22.22}$$

From this by Weakening we have $\Gamma_1 \Rightarrow^c \Delta_1$.

$\square$

Lemma 23. *The cut rule is admissible in* **Gc**.

$$\frac{\Gamma \Rightarrow^c s, \Delta \qquad s, \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \Rightarrow^c \Delta, \Delta'} \text{Cut}$$

Lemma 24. *If* $\Gamma \Rightarrow^c \Delta$ *then* $\Gamma, \neg\Delta \vdash^c \bot$, where $\neg\Delta$ is the set of negations of formulas in Δ.

Theorem 25 (Equivalence of **Nc** and **Gc**). $\Gamma \vdash^c s$ *iff* $\Gamma \Rightarrow^c s$.

Proof. From left to right: by induction on the derivation of $\Gamma \vdash^c s$. Cut is needed for **Gc**.

From right to left: follows from Lemma 24. ☐

Lemma 26 (Generalized cut for **Gi**). *The generalized cut rule is admissible in* **Gi**.

$$\frac{\Gamma \Rightarrow^i s \qquad \Gamma' \Rightarrow^i u}{\Gamma, \Gamma' \setminus s \Rightarrow^i u}\,\text{GCut}$$

Proof. By induction on the structure of the cut formula s (first), then on the derivation $\Gamma \Rightarrow^i s$ (second), and then on derivation $\Gamma' \Rightarrow^i u$ (third) if necessary. The base cases (variables and Falsehood) do not need the third induction. Most cases are simple and Coq with little guidance can check them automatically. The following cases need detailed inspection. In these cases, the 3rd induction is needed. The inductive hypothesis from the 2nd induction is not used. The inductive hypotheses from the 1st induction are the same for all of these cases:

$$\forall\, \Gamma\ \Gamma'\ u.\ \ \Gamma \Rightarrow^i s_1 \quad\longrightarrow\quad \Gamma' \Rightarrow^i u \quad\longrightarrow\quad \Gamma, (\Gamma' \setminus s_1) \Rightarrow^i u \tag{26.1}$$

$$\forall\, \Gamma\ \Gamma'\ u.\ \ \Gamma \Rightarrow^i s_2 \quad\longrightarrow\quad \Gamma' \Rightarrow^i u \quad\longrightarrow\quad \Gamma, (\Gamma' \setminus s_2) \Rightarrow^i u \tag{26.2}$$

(i) $s = s_1 \to s_2$

We have some $s' \to t' \in \Gamma'$, and $\Gamma' \Rightarrow^i s'$, and $t', \Gamma' \Rightarrow^i u$, and $s_1, \Gamma \Rightarrow^i s_2$.

The inductive hypotheses from the 3rd induction are

$$\Gamma, (\Gamma' \setminus s_1 \to s_2) \Rightarrow^i s' \tag{26.3}$$

$$\Gamma, ((t', \Gamma') \setminus s_1 \to s_2) \Rightarrow^i u \tag{26.4}$$

We have to show $\Gamma, (\Gamma' \setminus s_1 \to s_2) \Rightarrow^i u$.

- If $s_1 \to s_2 = s' \to t'$, i.e. $s_1 = s'$ and $s_2 = t'$, then from 26.3 and $s_1, \Gamma \Rightarrow^i s_2$, using 26.1 we have $\Gamma, (\Gamma' \setminus s_1 \to s_2), ((s_1, \Gamma) \setminus s_1) \Rightarrow^i s_2$, from which by Weakening we have

$$\Gamma, (\Gamma' \setminus s_1 \to s_2) \Rightarrow^i s_2 \tag{26.5}$$

  Now, from 26.5 and 26.4, using 26.2 we have

$$\Gamma, (\Gamma' \setminus s_1 \to s_2), ((\Gamma, ((s_2, \Gamma') \setminus s_1 \to s_2)) \setminus s_2) \Rightarrow^i u \tag{26.6}$$

  From this by Weakening again, we have $\Gamma, (\Gamma' \setminus s_1 \to s_2) \Rightarrow^i u$.
- If $s_1 \to s_2 \neq s' \to t'$, then $s' \to t' \in (\Gamma, (\Gamma' \setminus s_1 \to s_2))$, by $\to$L we need only to show $t', \Gamma, (\Gamma' \setminus s_1 \to s_2) \Rightarrow^i u$, which follows from Weakening of 26.4.

(ii) $s = s_1 \wedge s_2$

We have some $s' \wedge t' \in \Gamma'$, and $s', t', \Gamma' \Rightarrow^i u$, and $\Gamma \Rightarrow^i s_1$, and $\Gamma \Rightarrow^i s_2$.

The inductive hypothesis from the 3rd induction is

$$\Gamma, ((s', t', \Gamma') \setminus s_1 \wedge s_2) \Rightarrow^i u \tag{26.7}$$

We have to show $\Gamma, (\Gamma' \setminus s_1 \wedge s_2) \Rightarrow^i u$.

- If $s_1 \wedge s_2 = s' \wedge t'$, i.e. $s_1 = s'$ and $s_2 = t'$, then from $\Gamma \Rightarrow^i s_1$ and 26.7, using 26.1 we have $\Gamma, ((\Gamma, ((s_1, s_2, \Gamma') \setminus s_1 \wedge s_2)) \setminus s_1) \Rightarrow^i u$. With this and $\Gamma \Rightarrow^i s_2$, using 26.2, we have

$$\Gamma, ((\Gamma, ((\Gamma, ((s_1, s_2, \Gamma') \setminus s_1 \wedge s_2)) \setminus s_1)) \setminus s_2) \Rightarrow^i u \tag{26.8}$$

  from which by Weakening we have $\Gamma, (\Gamma' \setminus s_1 \wedge s_2) \Rightarrow^i u$.
- If $s_1 \wedge s_2 \neq s' \wedge t'$, then $s' \wedge t' \in (\Gamma, (\Gamma' \setminus s_1 \wedge s_2))$, by $\wedge$L we need only to show $s', t', \Gamma, (\Gamma' \setminus s_1 \wedge s_2) \Rightarrow^i u$, which follows from Weakening of 26.7.

(iii) $s = s_1 \vee s_2$

We have some $s' \vee t' \in \Gamma'$, and $s', \Gamma' \Rightarrow^i u$, and $t', \Gamma' \Rightarrow^i u$, and $\Gamma \Rightarrow^i s_1$ (there is another similar case where we have $\Gamma \Rightarrow^i s_2$ instead of $\Gamma \Rightarrow^i s_1$, in which the reasoning is the same).

The inductive hypotheses from the 3rd induction are

$$\Gamma, ((s', \Gamma') \setminus s_1 \vee s_2) \Rightarrow^i u \tag{26.9}$$

$$\Gamma, ((t', \Gamma') \setminus s_1 \vee s_2) \Rightarrow^i u \tag{26.10}$$

We have to show $\Gamma, (\Gamma' \setminus s_1 \vee s_2) \Rightarrow^i u$.

- If $s_1 \vee s_2 = s' \vee t'$, i.e. $s_1 = s'$ and $s_2 = t'$, with $\Gamma \Rightarrow^i s_1$ and 26.9, using 26.1 we have

$$\Gamma, ((\Gamma, ((s_1, \Gamma') \setminus s_1 \vee s_2)) \setminus s_1) \Rightarrow^i u \tag{26.11}$$

  from which by Weakening we have $\Gamma, (\Gamma' \setminus s_1 \vee s_2) \Rightarrow^i u$.
- If $s_1 \vee s_2 \neq s' \vee t'$, then $s' \vee t' \in (\Gamma, (\Gamma' \setminus s_1 \vee s_2))$, by $\vee$L we have to show $s', \Gamma, \Gamma' \setminus s_1 \vee s_2 \Rightarrow^i u$ and $t', \Gamma, \Gamma' \setminus s_1 \vee s_2 \Rightarrow^i u$, which follow from Weakening of 26.9 and 26.10, respectively.

□

Lemma 27. *The cut rule is admissible in* **Gi**.

$$\frac{\Gamma \Rightarrow^i s \qquad s, \Gamma \Rightarrow^i t}{\Gamma \Rightarrow^i t}\ \text{Cut}$$

Theorem 28 (Equivalence of **Ni** and **Gi**)**.** $\Gamma \vdash^i s$ *iff* $\Gamma \Rightarrow^i s$.

Theorem 29. *Intuitionistic sequent entailment is decidable.*

$$\frac{}{\Gamma \vdash_H^c s \to t \to s}\ \text{K} \qquad \frac{}{\Gamma \vdash_H^c (s \to t \to u) \to (s \to t) \to s \to u}\ \text{S}$$

$$\frac{}{\Gamma \vdash_H^c \neg\neg s \to s}\ \text{DN} \qquad \frac{}{\Gamma \vdash_H^c s \to t \to s \wedge t}\ \wedge\text{R}$$

$$\frac{}{\Gamma \vdash_H^c s \wedge t \to s}\ \wedge\text{L1} \qquad \frac{}{\Gamma \vdash_H^c s \wedge t \to t}\ \wedge\text{L2}$$

$$\frac{}{\Gamma \vdash_H^c s \to s \vee t}\ \vee\text{R1} \qquad \frac{}{\Gamma \vdash_H^c t \to s \vee t}\ \vee\text{R2}$$

$$\frac{}{\Gamma \vdash_H^c (s \to u) \to (t \to u) \to (s \vee t \to u)}\ \vee\text{L}$$

$$\frac{s \in \Gamma}{\Gamma \vdash_H^c s}\ \text{A} \qquad \frac{\Gamma \vdash_H^c s \to t \qquad \Gamma \vdash_H^c s}{\Gamma \vdash_H^c t}\ \text{MP}$$

Figure 5: Rules for the classical Hilbert system **Hc**

$$\frac{}{\Gamma \vdash_H^i s \to t \to s}\ \text{K} \qquad \frac{}{\Gamma \vdash_H^i (s \to t \to u) \to (s \to t) \to s \to u}\ \text{S}$$

$$\frac{}{\Gamma \vdash_H^i \bot \to s}\ \text{E} \qquad \frac{}{\Gamma \vdash_H^i s \to t \to s \wedge t}\ \wedge\text{R}$$

$$\frac{}{\Gamma \vdash_H^i s \wedge t \to s}\ \wedge\text{L1} \qquad \frac{}{\Gamma \vdash_H^i s \wedge t \to t}\ \wedge\text{L2}$$

$$\frac{}{\Gamma \vdash_H^i s \to s \vee t}\ \vee\text{R1} \qquad \frac{}{\Gamma \vdash_H^i t \to s \vee t}\ \vee\text{R2}$$

$$\frac{}{\Gamma \vdash_H^i (s \to u) \to (t \to u) \to (s \vee t \to u)}\ \vee\text{L}$$

$$\frac{s \in \Gamma}{\Gamma \vdash_H^i s}\ \text{A} \qquad \frac{\Gamma \vdash_H^i s \to t \qquad \Gamma \vdash_H^i s}{\Gamma \vdash_H^i t}\ \text{MP}$$

Figure 6: Rules for the intuitionistic Hilbert system **Hi**

4 Hilbert's systems

Definition 30 (Classical Hilbert system **Hc**)**.** The classical Hilbert system defines the entailment relation $\Gamma \vdash_H^c s$ by rules given in Figure 5.

Definition 31 (Intuitionistic Hilbert system **Hi**)**.** The intuitionistic sequent system defines the entailment relation $\Gamma \vdash_H^i s$ by rules given in Figure 6.

Lemma 32 (Deduction theorem for **Hc** and **Hi**)**.** *If* $s, \Gamma \vdash_H t$ *then* $\Gamma \vdash_H s \to t$.

Lemma 33 (Equivalence of **Hc** and **Nc**)**.** $\Gamma \vdash_H^c s$ *iff* $\Gamma \vdash^c s$.

Lemma 34 (Equivalence of **Hi** and **Ni**)**.** $\Gamma \vdash_H^i s$ *iff* $\Gamma \vdash^i s$.

5 Semantic tableaux

Definition 35 (Intuitionistic tableau system **F**)**.** The intuitionistic tableau system defines the entailment relation $\Gamma \Rightarrow_F \Delta$ by rules given in Figure 7. There is an interesting, close resemblance with the classical Gentzen system.

$$\frac{\bot \in \Gamma}{\Gamma \Rightarrow_F \Delta}\,\text{F} \qquad \frac{x \in \Gamma \qquad x \in \Delta}{\Gamma \Rightarrow_F \Delta}\,\text{C}$$

$$\frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow_F s, \Delta \qquad t, \Gamma \Rightarrow_F \Delta}{\Gamma \Rightarrow_F \Delta}\,{\to}\text{L} \qquad \frac{s \to t \in \Delta \qquad s, \Gamma \Rightarrow_F t}{\Gamma \Rightarrow_F \Delta}\,{\to}\text{R}$$

$$\frac{s \wedge t \in \Gamma \qquad s, t, \Gamma \Rightarrow_F \Delta}{\Gamma \Rightarrow_F \Delta}\,\wedge\text{L} \qquad \frac{s \wedge t \in \Delta \qquad \Gamma \Rightarrow_F s, \Delta \qquad \Gamma \Rightarrow_F t, \Delta}{\Gamma \Rightarrow_F \Delta}\,\wedge\text{R}$$

$$\frac{s \vee t \in \Gamma \qquad s, \Gamma \Rightarrow_F \Delta \qquad t, \Gamma \Rightarrow_F \Delta}{\Gamma \Rightarrow_F \Delta}\,\vee\text{L} \qquad \frac{s \vee t \in \Delta \qquad \Gamma \Rightarrow_F s, t, \Delta}{\Gamma \Rightarrow_F \Delta}\,\vee\text{R}$$

Figure 7: Rules for the intuitionistic tableau system **F**

Fact 36 (Reflexivity of **F**)**.** If $s \in \Gamma$ and $s \in \Delta$ then $\Gamma \Rightarrow_F \Delta$.

Lemma 37. *Weakening is admissible in* **F**.

$$\frac{\Gamma \Rightarrow_F \Delta \qquad \Gamma \subseteq \Gamma' \qquad \Delta \subseteq \Delta'}{\Gamma' \Rightarrow_F \Delta'}\,\text{Weak}$$

Lemma 38 (**F** to **Gi**)**.** *If* $\Gamma \Rightarrow_F \Delta$ *then* $\Gamma \Rightarrow^i \bigvee \Delta$.

Proof. By induction on $\Gamma \Rightarrow_F \Delta$. This observation is needed: if $\Gamma \subseteq \Delta$, then $\bigvee \Gamma \Rightarrow^i \bigvee \Delta$. □

Theorem 39 (Equivalence of **F** and **Gi**)**.** $\Gamma \Rightarrow_F s$ *iff* $\Gamma \Rightarrow^i s$ *and* $\Gamma \Rightarrow_F \Delta$ *iff* $\Gamma \Rightarrow^i \bigvee \Delta$.

Theorem 40. *Intuitionistic tableau entailment is decidable.*

Theorem 41 (Equivalence of **F** and **Ni**)**.** $\Gamma \Rightarrow_F s$ *iff* $\Gamma \vdash^i s$ *and* $\Gamma \Rightarrow_F \Delta$ *iff* $\Gamma \vdash^i \bigvee \Delta$.

Definition 42 ($\circ$-free)**.** A formula s is called $\circ$-free, where $\circ$ is a placeholder for some connective or constant of the logic, if $\circ$ does not appear in s. For example, that s is $\wedge$-free is defined recursively as:

(i) if s is a propositional variable, then s is $\wedge$-free

(ii) if $s = \bot$, then s is $\wedge$-free

(iii) if $s = s_1 \vee s_2$, then s is $\wedge$-free if s_1 and s_2 are $\wedge$-free

(iv) if $s = s_1 \to s_2$, then s is $\wedge$-free if s_1 and s_2 are $\wedge$-free.

Lemma 43. *If* Δ *is a set of* $\to$*-free formulas, and* $\Rightarrow^c \Delta$*, then* $\Rightarrow_F \Delta$.

Proof. By induction on $\Rightarrow^c \Delta$. □

Lemma 44. *If s is $\to$-free, then $\vdash^i s$ iff $\vdash^c s$.*

Proof. Follows from Lemma 10, Lemma 43, Theorem 25, and Theorem 39. □

6 Remarks

The systems represented here are based on those represented by Troelstra and Schwichtenberg [1996], except that the tableau system is the formulation of [Fitting, 1969], who called it Beth tableau. The original tableau system is by [Kripke, 1963]. All systems are adjusted to work with lists of formulas in the fashion of [Smolka and Brown, 2014]. The Gentzen systems here are closest to the **GK3**[**ci**] systems in [Troelstra and Schwichtenberg, 1996], which are due to Stephen Kleene. The cut-elimination proofs for Gentzen systems are based on [Smolka and Brown, 2014].

References

Melvin Fitting. *Intuitionistic Logic, Model Theory and Forcing.* Amsterdam, North-Holland Pub. Co., 1969.

Saul A. Kripke. Semantical analysis of intuitionistic logic I. In Michael Dummett and J. N. Crossley, editors, *Journal of Symbolic Logic*, volume 2, pages 92–130. North Holland, 1963.

G. Smolka and C. E. Brown. Introduction to computational logic, lecture notes. 2014. URL https://www.ps.uni-saarland.de/courses/cl-ss14/script/icl.pdf.

A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1996.