Propositional Logic revisited

Research Immersion Lab, Summer Break 2015

Hoang Hai Dang

Supervisor: Prof. Dr. Gert Smolka

Saarland University

October 23, 2015

Introduction

Starting Point

- ICL'14: intuitionistic Gentzen system, with decidability by proof search

Starting Point

- ICL'14: intuitionistic Gentzen system, with decidability by proof search
- ICL'13: clausal models, with decidability by tableau

Starting Point

- ICL'14: intuitionistic Gentzen system, with decidability by proof search
- ICL'13: clausal models, with decidability by tableau
- ICL'12: Kripke models and the tableau system

Starting Point

- ICL'14: intuitionistic Gentzen system, with decidability by proof search
- ICL'13: clausal models, with decidability by tableau
- ICL'12: Kripke models and the tableau system

Starting Point

- ICL'14: intuitionistic Gentzen system, with decidability by proof search
- ICL'13: clausal models, with decidability by tableau
- ICL'12: Kripke models and the tableau system

Task:

- Complete the logic with disjunction and conjunction
- Look at models for intutionistic logic and ...

Starting Point

- ICL'14: intuitionistic Gentzen system, with decidability by proof search
- ICL'13: clausal models, with decidability by tableau
- ICL'12: Kripke models and the tableau system

Task:

- Complete the logic with disjunction and conjunction
- Look at models for intutionistic logic and . . .

Result: various proofs in 5K+ lines of Coq - *lots of technical details*

What we will (very quickly) see

- **Cut elimination**: intuitionistic and classical Gentzen systems
- **Models for intutionistic logic**: Heyting algebras and Kripke models
  - **Countermodels**: independence of intuitionistic connectives
- **Constructing countermodels**: intuitionistic tableau and demos
- **Finding countermodels**: Naive Kripke models fail.

Cut Elimination

Cut Elimination for Gentzen's systems

$$\frac{\Gamma \Rightarrow^i s \qquad s, \Gamma \Rightarrow^i t}{\Gamma \Rightarrow^i t}\ \text{Cut}$$

- (Troelstra and Schwichtenberg 1996) proofs involve the definitions of *level*, *rank*, and *cutrank* of a cut.

Cut Elimination for Gentzen's systems

$$\frac{\Gamma \Rightarrow^i s \qquad s, \Gamma \Rightarrow^i t}{\Gamma \Rightarrow^i t} \text{ Cut}$$

- (Troelstra and Schwichtenberg 1996) proofs involve the definitions of *level*, *rank*, and *cutrank* of a cut.
- In ICL'14, the proof bases completely on nested structural inductions, by a generalisation:

$$\frac{\Gamma \Rightarrow^i s \qquad \Gamma' \Rightarrow^i u}{\Gamma, \Gamma' \setminus s \Rightarrow^i u} \text{ GCut}$$

A classical Gentzen system

$$\frac{x \in \Gamma \qquad x \in \Delta}{\Gamma \Rightarrow^c \Delta}\,\mathrm{A} \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow^c \Delta}\,\mathrm{E}$$

$$\frac{s \to t \in \Delta \qquad s, \Gamma \Rightarrow^c t, \Delta}{\Gamma \Rightarrow^c \Delta}\,{\to}\mathrm{R}$$

$$\frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow^c s, \Delta \qquad t, \Gamma \Rightarrow^c \Delta}{\Gamma \Rightarrow^c \Delta}\,{\to}\mathrm{L}$$

$$\frac{s \wedge t \in \Delta \qquad \Gamma \Rightarrow^c s, \Delta \qquad \Gamma \Rightarrow^c t, \Delta}{\Gamma \Rightarrow^c \Delta}\,{\wedge}\mathrm{R}$$

$$\frac{s \wedge t \in \Gamma \qquad s, t, \Gamma \Rightarrow^c \Delta}{\Gamma \Rightarrow^c \Delta}\,{\wedge}\mathrm{L}$$

...

Cut-elim for classical Gentzen

$$\frac{\Gamma \Rightarrow^c s, \Delta \qquad s, \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \Rightarrow^c \Delta, \Delta'}\ \mathrm{Cut}$$

Cut-elim for classical Gentzen

$$\frac{\Gamma \Rightarrow^c s, \Delta \qquad s, \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \Rightarrow^c \Delta, \Delta'} \text{Cut}$$

and the generalisation

$$\frac{\Gamma \Rightarrow^c \Delta \qquad \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \setminus s \Rightarrow^c \Delta \setminus s, \Delta'} \text{GCut}$$

Cut-elim for classical Gentzen

$$\frac{\Gamma \Rightarrow^c s, \Delta \qquad s, \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \Rightarrow^c \Delta, \Delta'} \text{ Cut}$$

and the generalisation

$$\frac{\Gamma \Rightarrow^c \Delta \qquad \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \setminus s \Rightarrow^c \Delta \setminus s, \Delta'} \text{ GCut}$$

This is actually also found in (Girard, Taylor, and Lafont 1989), and is closely related to the original generalisation of Gentzen:

$$\frac{\Gamma \Rightarrow^c \Delta, s^n \qquad s^m, \Gamma' \Rightarrow^c \Delta'}{\Gamma, \Gamma' \Rightarrow^c \Delta, \Delta'} \text{ Multicut}$$

Full of technical details, but routine

- The proof of the generalised cut for classical logic is much more complex than the intuitionistic case, but more *symmetric*.

Full of technical details, but routine

- The proof of the generalised cut for classical logic is much more complex than the intuitionistic case, but more *symmetric*.
- In one of the most obscure cases: find and prove the subgoal

$$\Gamma_1, (\Gamma_3 \setminus s), (\Gamma_1, \Gamma_2 \setminus s) \setminus t \Rightarrow^c (((\Delta_2 \setminus s), \Delta_3) \setminus t), (\Delta_2 \setminus s), \Delta_1$$

where

$$\begin{aligned}
\Gamma_1 &:= \Gamma, \Gamma' \setminus s \to t \\
\Delta_1 &:= (\Delta \setminus s \to t), \Delta' \\
\Gamma_2 &:= \Gamma, (t, \Gamma') \setminus s \to t \\
\Delta_2 &:= (\Delta \setminus s \to t), s, \Delta' \\
\Gamma_3 &:= s, \Gamma, \Gamma' \setminus s \to t \\
\Delta_3 &:= ((t, \Delta) \setminus s \to t), \Delta'
\end{aligned}$$

Heyting algebras and Kripke models

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

- Elements
- Truth values

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

• Elements	• Truth values
• A partial order $\leq$	• Ordering of truth values

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

- Elements
- A partial order $\leq$
- Bounded ordering

- Truth values
- Ordering of truth values
- Bounded truth values $\bot \leq x \leq \top$

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

- Elements
- A partial order $\leq$
- Bounded ordering
- Greatest lower bound

- Truth values
- Ordering of truth values
- Bounded truth values $\bot \leq x \leq \top$
- Truth value for $\wedge$

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

- Elements
- A partial order $\leq$
- Bounded ordering
- Greatest lower bound
- Least upper bound

- Truth values
- Ordering of truth values
- Bounded truth values $\bot \leq x \leq \top$
- Truth value for $\wedge$
- Truth value for $\vee$

Construction

Heyting algebras model **truth values** for formulas.

An *interpretation* maps a formula to a truth value.

- Elements
- A partial order $\leq$
- Bounded ordering
- Greatest lower bound
- Least upper bound
- A *mysterious concept*

- Truth values
- Ordering of truth values
- Bounded truth values $\bot \leq x \leq \top$
- Truth value for $\wedge$
- Truth value for $\vee$
- Truth value for $\rightarrow$

Heyting algebras

A Heyting algebra is a partial order $(H, \leq, \bot, \top, \wedge, \vee, \to)$ satisfying the following conditions for all $x, y, z \in H$:

- $\bot \leq x \leq \top$
- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \to y$ iff $z \wedge x \leq y$.

We can also define $\top = \bot \to \bot$.

Heyting algebras

A Heyting algebra is a partial order $(H, \leq, \bot, \top, \wedge, \vee, \rightarrow)$ satisfying the following conditions for all $x, y, z \in H$:

- $\bot \leq x \leq \top$
- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$.

We can also define $\top = \bot \rightarrow \bot$.

Completeness of Heyting algebras

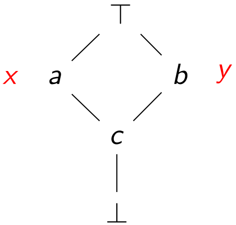
$\Gamma \Rightarrow^i s$ iff for any H and α, $\alpha(\Gamma) \leq \alpha s$.
Or s is intuitionistically derivable iff $\top \leq \alpha s$, for any H, α.

We proved this for *preordered* Heyting algebras partly basing on (Brown 2014). For *partial-ordered* algebras, (Troelstra and Dalen 1988) provided a proof with quotients (equivalence class).

Examples

- $z \le x \wedge y$ iff $z \le x$ and $z \le y$
- $x \vee y \le z$ iff $x \le z$ and $y \le z$
- $z \le x \to y$ iff $z \wedge x \le y$

With $\alpha x = a$ and $\alpha y = b$:

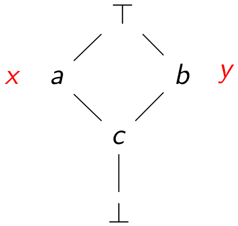


$\alpha(x \wedge \bot)$
$\alpha(x \vee \top)$
$\alpha(x \to \bot)$
$\alpha(x \to x)$
$\alpha(\bot \to \bot)$
$\alpha(x \wedge y)$
$\alpha(x \vee y)$
$\alpha(x \to y)$
$\alpha(x \vee \neg x)$
$\alpha(\neg\neg x)$
$\alpha(\neg\neg x \to x)$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

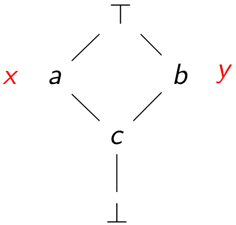


$\alpha(x \wedge \bot) = \bot$
$\alpha(x \vee \top)$
$\alpha(x \rightarrow \bot)$
$\alpha(x \rightarrow x)$
$\alpha(\bot \rightarrow \bot)$
$\alpha(x \wedge y)$
$\alpha(x \vee y)$
$\alpha(x \rightarrow y)$
$\alpha(x \vee \neg x)$
$\alpha(\neg\neg x)$
$\alpha(\neg\neg x \rightarrow x)$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

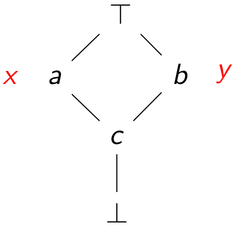


$$\begin{aligned}
&\alpha(x \wedge \bot) && = \bot \\
&\alpha(x \vee \top) && = \top \\
&\alpha(x \rightarrow \bot) && \\
&\alpha(x \rightarrow x) && \\
&\alpha(\bot \rightarrow \bot) && \\
&\alpha(x \wedge y) && \\
&\alpha(x \vee y) && \\
&\alpha(x \rightarrow y) && \\
&\alpha(x \vee \neg x) && \\
&\alpha(\neg\neg x) && \\
&\alpha(\neg\neg x \rightarrow x) &&
\end{aligned}$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \to y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

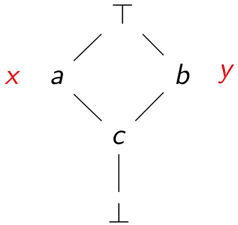


$$
\begin{aligned}
&\alpha(x \wedge \bot) && = \bot \\
&\alpha(x \vee \top) && = \top \\
&\alpha(x \to \bot) && = \bot \\
&\alpha(x \to x) && \\
&\alpha(\bot \to \bot) && \\
&\alpha(x \wedge y) && \\
&\alpha(x \vee y) && \\
&\alpha(x \to y) && \\
&\alpha(x \vee \neg x) && \\
&\alpha(\neg\neg x) && \\
&\alpha(\neg\neg x \to x) &&
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

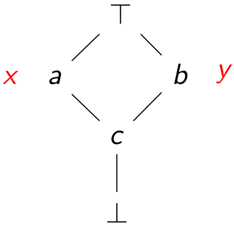


$$
\begin{aligned}
&\alpha(x \wedge \bot) && = \bot \\
&\alpha(x \vee \top) && = \top \\
&\alpha(x \rightarrow \bot) && = \bot \\
&\alpha(x \rightarrow x) && = \top \\
&\alpha(\bot \rightarrow \bot) && \\
&\alpha(x \wedge y) && \\
&\alpha(x \vee y) && \\
&\alpha(x \rightarrow y) && \\
&\alpha(x \vee \neg x) && \\
&\alpha(\neg\neg x) && \\
&\alpha(\neg\neg x \rightarrow x) &&
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

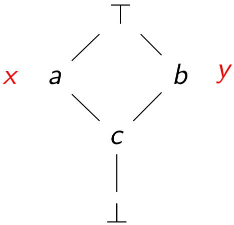


$$
\begin{aligned}
&\alpha(x \wedge \bot) && = \bot \\
&\alpha(x \vee \top) && = \top \\
&\alpha(x \rightarrow \bot) && = \bot \\
&\alpha(x \rightarrow x) && = \top \\
&\alpha(\bot \rightarrow \bot) && = \top \\
&\alpha(x \wedge y) && \\
&\alpha(x \vee y) && \\
&\alpha(x \rightarrow y) && \\
&\alpha(x \vee \neg x) && \\
&\alpha(\neg\neg x) && \\
&\alpha(\neg\neg x \rightarrow x) &&
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \to y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

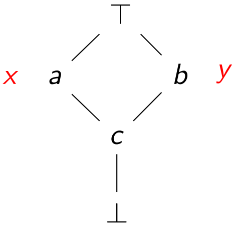


$$
\begin{aligned}
&\alpha(x \wedge \bot) && = \bot \\
&\alpha(x \vee \top) && = \top \\
&\alpha(x \to \bot) && = \bot \\
&\alpha(x \to x) && = \top \\
&\alpha(\bot \to \bot) && = \top \\
&\alpha(x \wedge y) && = c \\
&\alpha(x \vee y) && \\
&\alpha(x \to y) && \\
&\alpha(x \vee \neg x) && \\
&\alpha(\neg\neg x) && \\
&\alpha(\neg\neg x \to x) &&
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \to y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

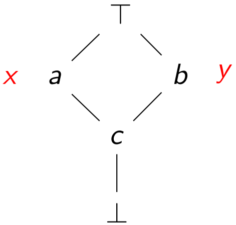


$$
\begin{aligned}
\alpha(x \wedge \bot) &= \bot \\
\alpha(x \vee \top) &= \top \\
\alpha(x \to \bot) &= \bot \\
\alpha(x \to x) &= \top \\
\alpha(\bot \to \bot) &= \top \\
\alpha(x \wedge y) &= c \\
\alpha(x \vee y) &= \top \\
\alpha(x \to y) & \\
\alpha(x \vee \neg x) & \\
\alpha(\neg\neg x) & \\
\alpha(\neg\neg x \to x) &
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \to y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

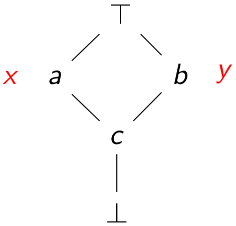


$$
\begin{aligned}
\alpha(x \wedge \bot) &= \bot \\
\alpha(x \vee \top) &= \top \\
\alpha(x \to \bot) &= \bot \\
\alpha(x \to x) &= \top \\
\alpha(\bot \to \bot) &= \top \\
\alpha(x \wedge y) &= c \\
\alpha(x \vee y) &= \top \\
\alpha(x \to y) &= b \\
\alpha(x \vee \neg x) & \\
\alpha(\neg\neg x) & \\
\alpha(\neg\neg x \to x) &
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \to y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

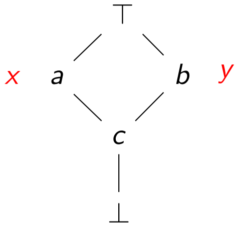


$$
\begin{aligned}
\alpha(x \wedge \bot) &= \bot \\
\alpha(x \vee \top) &= \top \\
\alpha(x \to \bot) &= \bot \\
\alpha(x \to x) &= \top \\
\alpha(\bot \to \bot) &= \top \\
\alpha(x \wedge y) &= c \\
\alpha(x \vee y) &= \top \\
\alpha(x \to y) &= b \\
\alpha(x \vee \neg x) &= a \\
\alpha(\neg\neg x) & \\
\alpha(\neg\neg x \to x) &
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:

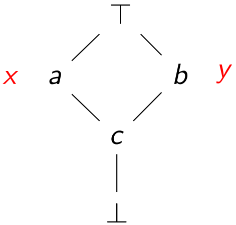


$$
\begin{aligned}
\alpha(x \wedge \bot) &= \bot \\
\alpha(x \vee \top) &= \top \\
\alpha(x \rightarrow \bot) &= \bot \\
\alpha(x \rightarrow x) &= \top \\
\alpha(\bot \rightarrow \bot) &= \top \\
\alpha(x \wedge y) &= c \\
\alpha(x \vee y) &= \top \\
\alpha(x \rightarrow y) &= b \\
\alpha(x \vee \neg x) &= a \\
\alpha(\neg\neg x) &= \bot \rightarrow \bot = \top \\
\alpha(\neg\neg x \rightarrow x) &
\end{aligned}
$$

Examples

- $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$
- $z \leq x \rightarrow y$ iff $z \wedge x \leq y$

With $\alpha x = a$ and $\alpha y = b$:



$$
\begin{aligned}
\alpha(x \wedge \bot) &= \bot \\
\alpha(x \vee \top) &= \top \\
\alpha(x \rightarrow \bot) &= \bot \\
\alpha(x \rightarrow x) &= \top \\
\alpha(\bot \rightarrow \bot) &= \top \\
\alpha(x \wedge y) &= c \\
\alpha(x \vee y) &= \top \\
\alpha(x \rightarrow y) &= b \\
\alpha(x \vee \neg x) &= a \\
\alpha(\neg\neg x) &= \bot \rightarrow \bot = \top \\
\alpha(\neg\neg x \rightarrow x) &= \top \rightarrow a = a
\end{aligned}
$$

Independence of Intuitionistic connectives

For classical logic, we can define $s \wedge t = \neg(\neg s \vee \neg t)$.

Lemma

Intuitionistic connectives are all independent, i.e. we cannot replace one connective with a construction of the remaining connectives.

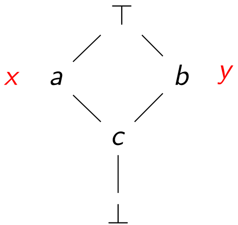
Independence of Intuitionistic connectives

For classical logic, we can define $s \wedge t = \neg(\neg s \vee \neg t)$.

Lemma

Intuitionistic connectives are all independent, i.e. we cannot replace one connective with a construction of the remaining connectives.

Proof by countermodels.



$\alpha(x \wedge y) = c$,
but any combination of x, y, $\bot$, $\top$, $\vee$ and $\rightarrow$ does not evaluate to c.
So there does not exist a combination that is equivalent to $x \wedge y$.

Kripke models for intuitionistic logic

A Kripke model[1] is a tuple $(K, \leq, \alpha)$ where:

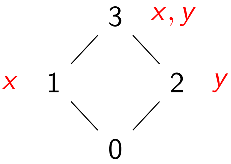
- $\leq$ is a preorder on the set of states K
- $\alpha : P \mapsto \mathcal{P}K$ is a monotonic labeling, meaning that if $p \in \alpha(x)$ and $p \leq q$ then $q \in \alpha(x)$.

[1] specialized model for intuitionistic logic

Kripke models for intuitionistic logic

A Kripke model[1] is a tuple $(K, \leq, \alpha)$ where:

- $\leq$ is a preorder on the set of states K
- $\alpha : P \mapsto \mathcal{P}K$ is a monotonic labeling, meaning that if $p \in \alpha(x)$ and $p \leq q$ then $q \in \alpha(x)$.



$$\alpha(x) = \{1, 3\}$$
$$\alpha(y) = \{2, 3\}$$

[1] specialized model for intuitionistic logic

Interpretation of a Kripke model

$$\hat{K}x := \alpha(x)$$
$$\hat{K}\bot := \emptyset$$
$$\hat{K}(s \wedge t) := \hat{K}s \cap \hat{K}t$$
$$\hat{K}(s \vee t) := \hat{K}s \cup \hat{K}t$$
$$\hat{K}(s \to t) := \{p \in K \mid (p\uparrow) \cap \hat{K}s \subseteq \hat{K}t\}$$

where $(p\uparrow) := \{q \in K \mid p \leq q\}$. $\hat{K}\emptyset := K$; $\hat{K}(s, \Gamma) := \hat{K}s \cap \hat{K}(\Gamma)$.

Read $\hat{K}s$ as the *set of states* in K that **force** or **satisfy** s.

Interpretation of a Kripke model

$$\hat{K}x := \alpha(x)$$
$$\hat{K}\bot := \emptyset$$
$$\hat{K}(s \wedge t) := \hat{K}s \cap \hat{K}t$$
$$\hat{K}(s \vee t) := \hat{K}s \cup \hat{K}t$$
$$\hat{K}(s \rightarrow t) := \{p \in K \mid (p\uparrow) \cap \hat{K}s \subseteq \hat{K}t\}$$

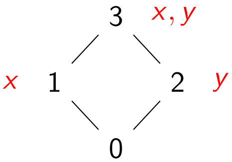
where $(p\uparrow) := \{q \in K \mid p \leq q\}$. $\hat{K}\emptyset := K$; $\hat{K}(s, \Gamma) := \hat{K}s \cap \hat{K}(\Gamma)$.

Read $\hat{K}s$ as the *set of states* in K that **force** or **satisfy** s.

Soundness of Kripke models

If $\Gamma \Rightarrow^{i} s$ then for any Kripke model K, $\hat{K}(\Gamma) \subseteq \hat{K}s$.
Or if s is intuitionistically derivable then $\hat{K}s = K$, for any K.

Examples



$$
\begin{aligned}
\hat{K}x &= \{1,3\} \\
\hat{K}y &= \{2,3\} \\
\hat{K}(x \wedge y) &= \{3\} \\
\hat{K}(x \vee y) &= \{1,2,3\} \\
\hat{K}(x \to y) &= \{2,3\} \\
\hat{K}(\neg x) &= \emptyset \\
\hat{K}(x \vee \neg x) &= \{1,3\} \\
\hat{K}(\neg\neg x \to x) &= \{1,3\}
\end{aligned}
$$

Kripke models to Heyting algebras

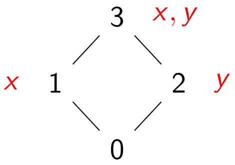
$$\text{upward-closed } A := \forall p \in A, p \leq q \longrightarrow q \in A$$
$$\overline{K} := \{A \mid A \subseteq K \wedge \text{upward-closed } A\}$$

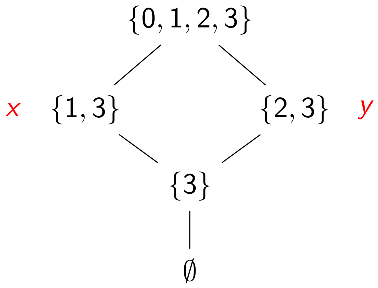
$(\overline{K}, \subseteq)$ is a Heyting algebra, with $\bot = \emptyset$, and $\top = K$, and the 3 operations:

$$A \wedge B := A \cap B$$
$$A \vee B := A \cup B$$
$$A \to B := \{p \in K \mid (p \uparrow) \cap A \subseteq B\}$$

The interpretation $\alpha(x) := \hat{K}x$. By induction $\alpha(s) = \hat{K}s$.

Examples





$$\alpha(x) = \{1, 3\}$$
$$\alpha(y) = \{2, 3\}$$
$$\alpha(x \wedge y) = \{3\}$$
$$\alpha(x \vee y) = \{1, 2, 3\}$$
$$\alpha(x \to y) = \{2, 3\}$$
$$\alpha(\neg x) = \emptyset$$
$$\alpha(x \vee \neg x) = \{1, 3\}$$
$$\alpha(\neg\neg x \to x) = \{1, 3\}$$

Intuitionistic tableau

Tableau system

- Decision procedure that produces counter Kripke models, first by (Kripke 1963), reformalized compactly by (Fitting 1969).

Tableau system

- Decision procedure that produces counter Kripke models, first by (Kripke 1963), reformalized compactly by (Fitting 1969).
- Has the same rules as the **classical** Gentzen system except the *negative implication* (right implication).

Tableau system

- Decision procedure that produces counter Kripke models, first by (Kripke 1963), reformalized compactly by (Fitting 1969).

- Has the same rules as the **classical** Gentzen system except the *negative implication* (right implication).

$$\frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow_F s, \Delta \qquad t, \Gamma \Rightarrow_F \Delta}{\Gamma \Rightarrow_F \Delta} \to\mathrm{L}$$

$$\frac{s \to t \in \Delta \qquad s, \Gamma \Rightarrow_F t, \cancel{\Delta}}{\Gamma \Rightarrow_F \Delta} \to\mathrm{R}$$

$$\dots$$

Tableau system

- Decision procedure that produces counter Kripke models, first by (Kripke 1963), reformalized compactly by (Fitting 1969).

- Has the same rules as the **classical** Gentzen system except the *negative implication* (right implication).

$$\frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow_F s, \Delta \qquad t, \Gamma \Rightarrow_F \Delta}{\Gamma \Rightarrow_F \Delta} \to\mathrm{L}$$

$$\frac{s \to t \in \Delta \qquad s, \Gamma \Rightarrow_F t, \not{\Delta}}{\Gamma \Rightarrow_F \Delta} \to\mathrm{R}$$

$$\cdots$$

Completeness of Tableau

$\Gamma \Rightarrow_F \Delta$ iff $\Gamma \Rightarrow^i \bigvee \Delta$

Signed formulation by Fitting

$$\frac{C, s \wedge t^{+}}{C, s^{+}, t^{+}} \wedge^{+} \qquad \frac{C, s \wedge t^{-}}{C, s^{-} \mid C, t^{-}} \wedge^{-}$$

$$\frac{C, s \vee t^{+}}{C, s^{+} \mid C, t^{+}} \vee^{+} \qquad \frac{C, s \vee t^{-}}{C, s^{-}, t^{-}} \vee^{-}$$

$$\frac{C, s \to t^{+}}{C, s^{-} \mid C, t^{+}} \to^{+} \qquad \frac{C, s \to t^{-}}{C^{+}, s^{+}, t^{-}} \to^{-}$$

Signed formulation by Fitting

$$\frac{C, s \wedge t^+}{C, s^+, t^+} \wedge^+ \qquad \frac{C, s \wedge t^-}{C, s^- \mid C, t^-} \wedge^-$$

$$\frac{C, s \vee t^+}{C, s^+ \mid C, t^+} \vee^+ \qquad \frac{C, s \vee t^-}{C, s^-, t^-} \vee^-$$

$$\frac{C, s \to t^+}{C, s^- \mid C, t^+} \to^+ \qquad \frac{C, s \to t^-}{C^+, s^+, t^-} \to^-$$

If s is **not** intuitionistically derivable, then there is one tableau of $[s^-]$ that is **not** *closed*. That tableau is the counter Kripke model.

We call such countermodel a **Demo**.

Examples

- The tableau of $x \to \neg\neg x$ is closed

Examples

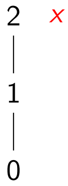
- The tableau of $x \to \neg\neg x$ is closed
- The tableau of $\neg\neg x \to x$ results in a demo:

$$\neg\neg x \to x^-$$
$$\downarrow$$
$$\neg\neg x^+, x^-, \neg x^-$$
$$\downarrow$$
$$\neg\neg x^+, x^+, \bot^-$$

Examples

- The tableau of $x \to \neg\neg x$ is closed
- The tableau of $\neg\neg x \to x$ results in a demo:

$$\neg\neg x \to x^{-}$$
$$\downarrow$$
$$\neg\neg x^{+}, x^{-}, \neg x^{-}$$
$$\downarrow$$
$$\neg\neg x^{+}, x^{+}, \bot^{-}$$



Examples

- The tableau of $x \to \neg\neg x$ is closed
- The tableau of $\neg\neg x \to x$ results in a demo:

$$\begin{array}{c} \neg\neg x \to x^- \\ \downarrow \\ \neg\neg x^+, x^-, \neg x^- \\ \downarrow \\ \neg\neg x^+, x^+, \bot^- \end{array} \qquad \begin{array}{c} 2 \quad {\color{red}x} \\ | \\ 1 \\ | \\ 0 \end{array} \qquad \begin{array}{r} \hat{K}x = \{2\} \\ \hat{K}(\neg x) = \emptyset \\ \hat{K}(\neg\neg x) = \{0, 1, 2\} \\ \hat{K}(\neg\neg x \to x) = \{2\} \neq K \end{array}$$

Observations

- Terminal nodes in a demo act **classically**

Observations

- Terminal nodes in a demo act **classically**
- The **jump** rule of the negative implication creates the non-terminal nodes, thus the intuitionistic sense:

Observations

- Terminal nodes in a demo act **classically**
- The **jump** rule of the negative implication creates the non-terminal nodes, thus the intuitionistic sense:

Fact

If Δ is $\rightarrow$-free, then $\Rightarrow^c \Delta$ iff $\Rightarrow_F \Delta$.
Or if s is $\rightarrow$-free, then $\Rightarrow^i s$ iff $\Rightarrow^c s$.

Observations

- Terminal nodes in a demo act **classically**
- The **jump** rule of the negative implication creates the non-terminal nodes, thus the intuitionistic sense:

Fact

If Δ is $\rightarrow$-free, then $\Rightarrow^c \Delta$ iff $\Rightarrow_F \Delta$.
Or if s is $\rightarrow$-free, then $\Rightarrow^i s$ iff $\Rightarrow^c s$.

Fact

If the set of signed subformulas of s^- generated by the tableau rules does not contain negative implications, then $\Rightarrow^i s$ iff $\Rightarrow^c s$.

Naive Kripke models

The naive model conjecture

- A demo is a counter Kripke model with states as sets of signed subformulas

The naive model conjecture

- A demo is a counter Kripke model with states as sets of signed subformulas
- Looking at countermodels for known underivability results:

Conjecture

If s is not intuitionistically derivable, there exists a counter Kripke model with states as sets of *only* (subformula) **positive variables**.

The naive model conjecture

- A demo is a counter Kripke model with states as sets of signed subformulas
- Looking at countermodels for known underivability results:

Conjecture

If s is not intuitionistically derivable, there exists a counter Kripke model with states as sets of *only* (subformula) **positive variables**.

- The conjecture was rejected by counterexamples found by a computer program:
  - $\neg x \vee \neg\neg x$
  - $\neg x \vee \neg x \rightarrow x$
  - $\neg x \vee \neg x \rightarrow y$
  - $x \rightarrow y \vee \neg x \rightarrow y$
  - $x \rightarrow y \vee (x \rightarrow y) \rightarrow y$

The amended lemma

- The set of maximal consistent (underivable) clauses, with *positive formula subset* relation, is a **demo**

The amended lemma

- The set of maximal consistent (underivable) clauses, with *positive formula subset* relation, is a **demo**

Lemma

A maximal consistent (subformula) clause is identified by its positive variables and negative implications.

The amended lemma

- The set of maximal consistent (underivable) clauses, with *positive formula subset* relation, is a **demo**

Lemma

A maximal consistent (subformula) clause is identified by its positive variables and negative implications.

- Let us look again at the counterexample $\neg x \vee \neg\neg x$:

The amended lemma

- The set of maximal consistent (underivable) clauses, with *positive formula subset* relation, is a **demo**

Lemma

A maximal consistent (subformula) clause is identified by its positive variables and negative implications.

- Let us look again at the counterexample $\neg x \vee \neg\neg x$:

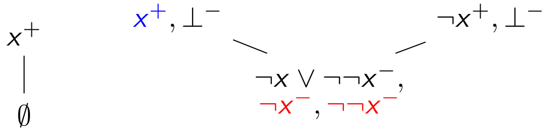
The amended lemma

- The set of maximal consistent (underivable) clauses, with *positive formula subset* relation, is a **demo**

Lemma

A maximal consistent (subformula) clause is identified by its positive variables and negative implications.

- Let us look again at the counterexample $\neg x \vee \neg\neg x$:



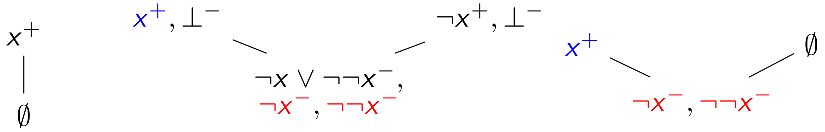
The amended lemma

- The set of maximal consistent (underivable) clauses, with *positive formula subset* relation, is a **demo**

Lemma

A maximal consistent (subformula) clause is identified by its positive variables and negative implications.

- Let us look again at the counterexample $\neg x \vee \neg\neg x$:



Recap

- Structural Cut elimination by generalisation originally from Gentzen

Recap

- Structural Cut elimination by generalisation originally from Gentzen
- Heyting algebras as complete semantics for intuitionistic propositional logic, obtainable from Kripke models

Recap

- Structural Cut elimination by generalisation originally from Gentzen
- Heyting algebras as complete semantics for intuitionistic propositional logic, obtainable from Kripke models
- The intuitionistic tableau provides a decision procedure by countermodels. Negative implications create the intuitionistic sense

Recap

- Structural Cut elimination by generalisation originally from Gentzen
- Heyting algebras as complete semantics for intuitionistic propositional logic, obtainable from Kripke models
- The intuitionistic tableau provides a decision procedure by countermodels. Negative implications create the intuitionistic sense
- Countermodels can be searched from the powerset of positive variable and negative implication subformulas.

References

Brown, C. E. 2014. "Semantics of Intuitionistic Propositional Logic: Heyting Algebras and Kripke Models." https://www.ps.uni-saarland.de/HeytingKripke/description.html.

Fitting, Melvin. 1969. *Intuitionistic Logic, Model Theory and Forcing*. Amsterdam, North-Holland Pub. Co.

Girard, Jean-Yves, Paul Taylor, and Yves Lafont. 1989. *Proofs and Types*. Cambridge University Press.

Kripke, Saul A. 1963. "Semantical Analysis of Intuitionistic Logic I." In *Journal of Symbolic Logic*, edited by Michael Dummett and J. N. Crossley, 2:92–130. North Holland.

Troelstra, A.S., and D. Dalen. 1988. *Constructivism in Mathematics: An Introduction*. Vol. 2. North-Holland.

Troelstra, A.S., and H. Schwichtenberg. 1996. *Basic Proof Theory*. Cambridge University Press.