Generating Infrastructural Code for Terms with Binders using MetaCoq

Bachelor Talk 1

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Motivation

- Prove metatheorems (e.g. preservation, normalization) of programming languages modelled in Coq
- Reason about binders and substitution

  $(\lambda x.t)v \succ_\beta t[x \mapsto v]$

Problem

How do we model binders? (i.e. form of the λ constructor)

Solution

Autosubst (dissertation of Kathrin Stark [Stark, 2019])

Different Approaches

Named	[Barendregt, 1984]
Locally Nameless	[McBride and McKinna, 2004]
Anti-Locally Nameless	[Laurent, 2021]
De Bruijn Indices	[De Bruijn, 1972]
Co De Bruijn Indices	[McBride, 2018]
HOAS	[Pfenning and Elliott, 1988]
⋮	⋮

Binders & Substitutions

Based on Sigma Calculus [Abadi et al., 1991]

- De Bruijn indices

$$\lambda x.\lambda y.x \cong \lambda.\lambda.1$$

- Parallel substitutions replace all variables

$$(\lambda.0\ 1)\ 3 \succ_\beta (0\ 1)[3 \cdot id] \succ_\sigma 3\ 0$$

$$[3 \cdot id] \cong [0 \mapsto 3; 1 \mapsto 0; \ldots; n+1 \mapsto n; \ldots]$$

- Confluent and terminating rewriting system

$\to$ Can decide whether $s[\sigma] = t[\tau]$ ([Schäfer et al., 2015])

Substitution Lemmas

Prove any assumption-free substitution lemma[1]

Metatheorem

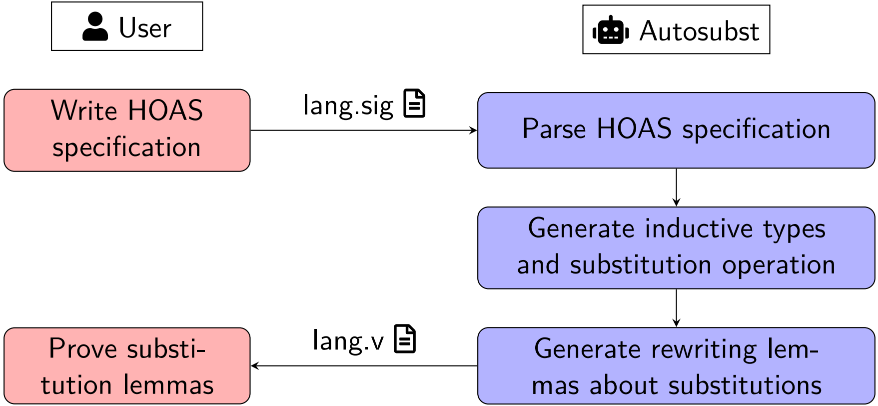
```
Lemma step_inst (σ: ℕ → tm) (s t: tm) :
  s ≻ t → s[σ] ≻ t[σ].
```

Goal in Proof

$$s[t \cdot id][\sigma] = s[\Uparrow \sigma][(t[\sigma]) \cdot id]$$
$$\Rightarrow s[t[\sigma] \cdot \sigma] = s[t[\sigma] \cdot \sigma]$$

[1] using functional extensionality

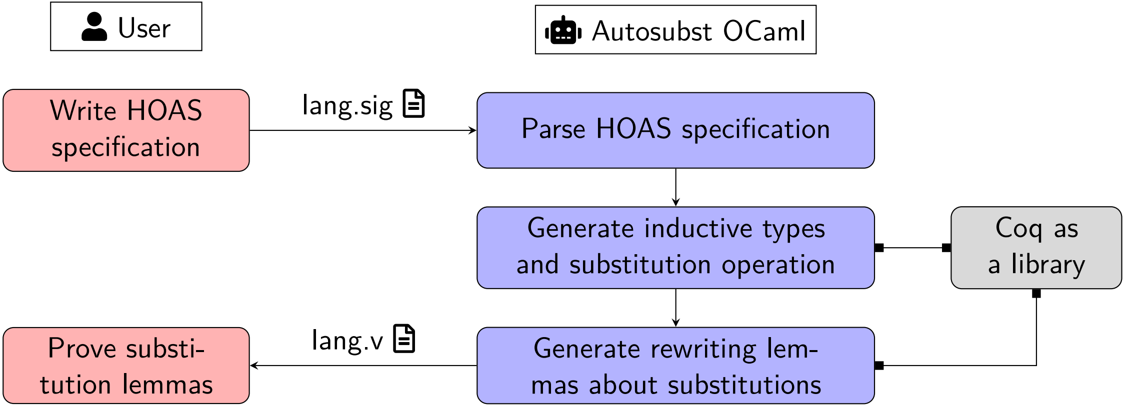
Workflow: Original Autosubst



Autosubst OCaml

- Reimplementation to use Coq as a library
- Use Coq AST and pretty printer to generate parseable code
- Tighter integration with Coq ecosystem

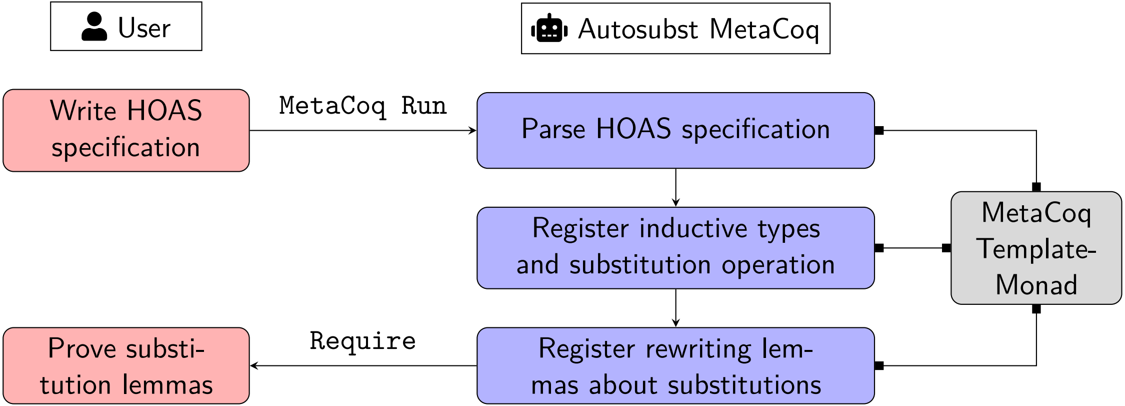
Workflow: Autosubst OCaml



Autosubst MetaCoq

- Reimplementation in MetaCoq
- Seamless usage from within Coq
- Even tighter integration with Coq ecosystem
- Correctness proofs might be possible

Workflow: Autosubst MetaCoq



HOAS [Pfenning and Elliott, 1988]

Used purely as notation for specifying syntax

Simply Typed Lambda Calculus

```
ty : Type
tm : Type

base : ty
arr  : ty → ty → ty

app : tm → tm → tm
lam : ty → (tm → tm) → tm
```

HOAS Parsing 1

(Custom Entry) Notations

```
Definition stlc_ctors :=
    {{ base : ty
       arr : ty → ty → ty;
       app : tm → tm → tm;
       lam : ty → (tm → tm) → tm }}.
```

- Deep embedding of HOAS
- Easy to use
- String handling behind the scenes is “unpalatable” ([Pit-Claudel and Bourgeat,])

HOAS Parsing 2

Parse Inductives with MetaCoq

```
Inductive tm :=
| app : tm → tm → tm
| lam : ty → (tm ↦ tm) → tm
```

- Need to know which inductive types are mutually inductive
- Pairs instead of negative occurrence

```
Notation "A ↦ B" := A * B
```

State Handling

Problem: AST uses de Bruijn indices and is Globalized

```
a = b ≅
(tApp (tInd {| inductive_mind :=
                (MPfile ["Logic"; "Init"; "Coq"], "eq"); ... |}
    [ tInd {| inductive_mind :=
               (MPfile ["Datatypes"; "Init"; "Coq"], "ℕ"); ... |}
    ; tRel 1
    ; tRel 0 ]))
```

Solution: Environments

$$dbmap : string \to \mathtt{M}\ \mathbb{N}$$

$$env : string \to \mathtt{M}\ term$$

MetaCoq Problems

Conceptual

- Defining tactics in MetaCoq
- Serialization of generated lemmas

Practical

- Term explosions due to laziness
- Debugging Gallina/MetaCoq programs

Extensions

Remove Functional Extensionality

Setoid rewriting should be possible

```
Lemma subst_tm_ext (s: tm) (σ τ: ℕ → tm):
  (∀ n, σ n = τ n) →
    s[σ] = s[τ].
```

New Lemmas

```
(* evaluating predicates on all free variables *)
Lemma allfv_term_impl (s: tm) (p q: ℕ → ℙ):
  (∀ n, p n → q n) →
    allfv_term p s → allfv_term q s.
```

Summary

	Original Autosubst	Autosubst OCaml	Autosubst MetaCoq
👍	fully featured	Coq library takes care of AST/printing	used from within Coq
👎	generated code less readable custom AST	coupled to Coq version no modular syntax	worse discoverability no modular syntax
Input	lang.sig 🗎	lang.sig 🗎	`Definition lang :=` </> `Inductive lang :=` </>
Output	lang.v 🗎	lang.v 🗎	Side effect that registers lemmas in Coq

Questions?

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Binders & Substitutions

Variables as Strings

```
| lambda : string → term → term
```

Natural but need to avoid capture $(\lambda x.y)[y \mapsto x] = \lambda x.x$
α-equivalence often just assumed

Single Substitutions

```
Lemma swap_subst: ∀ t x1 x2 v1 v2,
    x1 ≠ x2 →
    closed v1 → closed v2 →
    t[x1 ↦ v1][x2 ↦ v2] = t[x2 ↦ v2][x1 ↦ v1].
```

Simple but need to deal with multiple substitutions

Sigma Calculus Background

Renamings & Substitutions

$$\xi : \mathbb{N} \to \mathbb{N}$$
$$\uparrow n = n + 1$$
$$\sigma : \mathbb{N} \to \mathit{term}$$

$$x[\sigma] = \sigma\ x$$
$$(t_1\ \ t_2)[\sigma] = t_1[\sigma]\ \ t_2[\sigma]$$
$$(\lambda.t)[\sigma] = \lambda.t[\Uparrow \sigma]$$
$$\Uparrow \sigma = 0 \cdot (\sigma \circ \langle \uparrow \rangle)$$

Using parallel subsitutitons

$$\begin{aligned}(\lambda.(\lambda.0\ 1\ \ 2))3 &\to (\lambda.0\ 1\ \ 2)[3 \cdot \mathit{id}] \\ &= \lambda.(0\ 1\ \ 2)[\Uparrow (3 \cdot \mathit{id})] \\ &= \lambda.(0\ 1\ \ 2)[0 \cdot (3 \cdot \mathit{id} \circ [\uparrow])] \\ &= \lambda.0\ 4\ 1\end{aligned}$$

Sigma Calculus Background

Equational Theory of λ calculus

$$s[var] = s$$
$$s[\sigma][\tau] = s[\sigma \circ [\tau]]$$
$$var \circ [\sigma] \equiv \sigma$$
$$(\sigma \circ [\tau]) \circ [\theta] \equiv \sigma \circ [\tau \circ [\theta]]$$
$$\sigma \circ [var] \equiv \sigma$$

$$id \circ f \equiv f$$
$$f \circ id \equiv f$$
$$(f \circ g) \circ h \equiv f \circ (g \circ h)$$
$$(s \cdot \sigma) \circ f \equiv (f\ s) \cdot (\sigma \circ f)$$
$$\uparrow \circ (s \cdot \sigma) \equiv \sigma$$
$$0 \cdot \uparrow \equiv id$$
$$(\sigma\ 0) \cdot (\uparrow \circ \sigma) \equiv \sigma$$

Scope Variables in AST

Original Autosubst had a dedicated AST node for scope variables

Scoped Simply Typed Lambda Calculus

```
Inductive tm (n : ℕ) :=
  | var : fin n → tm n
  | app : tm n → tm n → tm n
  | lam : ty → tm (S n) → tm n.

tm n ≅ TmApp (TmId "tm") (SubstScope [TmId "n"])
     ≅ CApp (CRef "tm") [CRef "n"]
```

- Original Autosubst: don't print the scope variable nodes
- Autosubst OCaml: don't create nodes for scope variables

State Handling

Creating a Lemma

```
Definition make_newlemma : M term :=
  let xs := ["x0"; "x1"] in
  dbmap_adds xs;;
  ...
  oldlemma <- env_get "oldlemma" in
  ...
  let newlemma := build (dbmap_get "x1", oldlemma) in
  return ("newlemma", newlemma).

Definition tmNewlemma env : TemplateMonad Env :=
  (name, lemma) <- M.run make_newlemma env
  tmDefinition name lemma
  let nextenv := env ++ tmLookup name
  tmReturn nextenv.
```

Why Use Functional Extensionality

$$\forall s. s[\sigma][\tau] = s[\sigma \circ [\tau]]$$
$$[\sigma] \circ [\tau] = [\sigma \circ [\tau]]$$
$$\ldots = s[\ldots[\sigma] \circ [\tau] \ldots]$$

```
Lemma subst_tm_ext (s: tm) (σ τ: ℕ → tm):
  (∀ n, σ n = τ n) →
    s[σ] = s[τ].
```

Case Studies that (almost) use Autosubst

- Call by Push Value
- Completeness Theorems for First-Order Logic Analysed in Constructive Type Theory
- Coq à la carte

- Did not use it because of FE:
- Trakhtenbrot's Theorem in Coq
- Synthetic Undecidability and Incompleteness of First-Order Axiom Systems in Coq

Extensions

Traced Syntax ([Herbelin and Lee,])

```
Inductive tm (vs : list ℕ) :=
  | var : v → (H: v ∈ vs) → tm vs
  | app : tm vs → tm vs' → tm (vs ++ vs')
  | lam : ty → tm ⇑vs → tm vs.
```

Extensions

New Lemmas

```
Lemma allfv_term_impl (s: tm) (p q: ℕ → ℙ):
  (∀ n, p n → q n) →
    allfv_term p s → allfv_term q s.

Lemma subst_tm_ext (s: tm) (σ τ: ℕ → tm):
  (∀ n, σ n = τ n) →
    s[σ] = s[τ].

Lemma ext_allfv_subst_term (s: tm) (σ τ: ℕ → tm):
  allfv_term (fun x ⇒ σ x = τ x) s →
    s[σ] = s[τ].
```