Formalizing Stream-Calculus in Coq

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$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 \\
a_{n+2} &= a_n + a_{n+1}
\end{aligned}
$$

Fibonacci sequence as example

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_{n+2} &= F_n + F_{n+1}
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Fibonacci sequence as example

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_{n+2} &= F_n + F_{n+1} \end{aligned}$$

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n)$$

From Sequences to Streams

$$t_0, t_1, t_2, t_3, t_4 \ldots$$

From Sequences to Streams

$$t_0, t_1, t_2, t_3, t_4 \ldots$$

Interpretation as Sequence

$a_n := t_n$

From Sequences to Streams

$$t_0, t_1, t_2, t_3, t_4 \ldots$$

Interpretation as Sequence

$a_n := t_n$

Interpretation as Stream

head: $s_0 := t_0$
tail: $s' := t_1, t_2, t_3, t_4, \ldots$

From Sequences to Streams

$$1, 1, 1, 1, 1 \ldots$$

Interpretation as Sequence

$a_n := 1$

From Sequences to Streams

$$1, 1, 1, 1, 1 \ldots$$

Interpretation as Sequence

$a_0 := 1$

$a_{n+1} := a_n$

From Sequences to Streams

$$1, 1, 1, 1, 1 \ldots$$

Interpretation as Sequence

$a_0 := 1$
$a_{n+1} := a_n$

Interpretation as Stream

head: $s_0 := 1$
tail: $s' := s$

From Sequences to Streams

$$1, 1, 1, 1, 1 \ldots$$

Interpretation as Sequence

$a_0 := 1$
$a_{n+1} := a_n$
Recursion

Interpretation as Stream

head: $s_0 := 1$
tail: $s' := s$
Differential Equation

Addition

Adding Sequences

$(a + b)_n := a_n + b_n$

Addition

Adding Sequences

$(a+b)_n := a_n + b_n$

Adding Streams

$(r+s)_0 := r_0 + s_0$
$(r+s)' := r' + s'$

Addition

Adding Sequences

$(a + b)_n := a_n + b_n$

Ascending Sequence

$a_0 := 0$
$a_{n+1} := a_n + 1$

Adding Streams

$(r + s)_0 := r_0 + s_0$
$(r + s)' := r' + s'$

Addition

Adding Sequences

$(a + b)_n := a_n + b_n$

Adding Streams

$(r + s)_0 := r_0 + s_0$
$(r + s)' := r' + s'$

Ascending Sequence

$a_0 := 0$
$a_{n+1} := a_n + 1$

Ascending Stream

$s_0 := 0$
$s' := s + (1, 1, \ldots)$

Multiplication on Streams

Some Definitions

$0_0 := 0$
$0' := 0$
$[t] := (t, 0, 0, \dots)$
$[t]_0 := t$
$[t]' := 0$

Multiplication on Streams

Some Definitions

$0_0 := 0$
$0' := 0$
$[t] := (t, 0, 0, \dots)$
$[t]_0 := t$
$[t]' := 0$

Multiplying Streams

$(r \times s)_0 := r_0 \times s_0$
$(r \times s)' := [r_0] \times s' + r' \times s$

Multiplication on Streams

Some Definitions

$0_0 := 0$
$0' := 0$
$[t] := (t, 0, 0, \ldots)$
$[t]_0 := t$
$[t]' := 0$

Multiplying Streams

$(r \times s)_0 := r_0 \times s_0$
$(r \times s)' := [r_0] \times s' + r' \times s$

Behaviour of Multiplication

$0 \times s = 0$

$([t] \times s)_0 := t \times s_0$
$([t] \times s)' := [t] \times s' + 0 \times s$

$[1] \times s = s$

Multiplication on Streams

Some Definitions

$0_0 := 0$
$0' := 0$
$[t] := (t, 0, 0, \dots)$
$[t]_0 := t$
$[t]' := 0$

Multiplying Streams

$(r \times s)_0 := r_0 \times s_0$
$(r \times s)' := [r_0] \times s' + r' \times s$

Behaviour of Multiplication

$0 \times s = 0$

$([t] \times s)_0 := t \times s_0$
$([t] \times s)' := [t] \times s' + 0 \times s$

$[1] \times s = s$

Multiplying Polynoms

$a = a_0 x^0 + a_1 x^1 + \dots$
$b = b_0 x^0 + b_1 x^1 + \dots$
$a \times b = c_0 x^0 + c_1 x^1 + \dots$

$c_i = \sum_{l+k=i} a_l \times b_k$
$= a_0 \times b_i + \sum_{l+k=i-1} a_{l+1} \times b_k$

Backshift X

$$X = (0, 1, 0, 0, \dots)$$
$$X_0 := 0$$
$$X' := [1]$$

Backshift X

$$X = (0, 1, 0, 0, \dots)$$
$$X_0 := 0$$
$$X' := [1]$$

$$(X \times s)_0 = 0$$
$$(X \times s)' = X_0 \times s' + X' \times s = s$$

Backshift X

$$X = (0, 1, 0, 0, \dots)$$
$$X_0 := 0$$
$$X' := [1]$$

$$(X \times s)_0 = 0$$
$$(X \times s)' = X_0 \times s' + X' \times s = s$$

$$s = [s_0] + X \times s'$$

Division on Streams

Dividing Streams

$(s \times s^{-1}) = 1$

Division on Streams

Dividing Streams

$(s \times s^{-1}) = 1$

- $1 = (s \times s^{-1})_0 = s_0 \times s_0^{-1}$
  $\Rightarrow s_0^{-1} = 1/s_0$

Division on Streams

Dividing Streams

$(s \times s^{-1}) = 1$

- $1 = (s \times s^{-1})_0 = s_0 \times s_0^{-1}$
  $\Rightarrow s_0^{-1} = 1/s_0$
- $0 = (s \times s^{-1})' = [s_0] \times (s^{-1})' + s' \times s^{-1}$
  $\Rightarrow (s^{-1})' = -s' \times s^{-1} \times [s_0]^{-1}$

Division on Streams

Dividing Streams

$(s \times s^{-1}) = 1$

- $1 = (s \times s^{-1})_0 = s_0 \times s_0^{-1}$
  $\Rightarrow s_0^{-1} = 1/s_0$
- $0 = (s \times s^{-1})' = [s_0] \times (s^{-1})' + s' \times s^{-1}$
  $\Rightarrow (s^{-1})' = -s' \times s^{-1} \times [s_0]^{-1}$

Dividing Sequences

$$s_n^{-1} = \frac{-\sum_{l+k=n-1} s_{l+1} \times s_k^{-1}}{s_0}$$

Division on Streams

Dividing Streams

$(s \times s^{-1}) = 1$

- $1 = (s \times s^{-1})_0 = s_0 \times s_0^{-1}$
  $\Rightarrow s_0^{-1} = 1/s_0$
- $0 = (s \times s^{-1})' = [s_0] \times (s^{-1})' + s' \times s^{-1}$
  $\Rightarrow (s^{-1})' = -s' \times s^{-1} \times [s_0]^{-1}$

Dividing Sequences

$$s_n^{-1} = \frac{-\sum_{l+k=n-1} s_{l+1} \times s_k^{-1}}{s_0}$$

Constant Stream by Division

$(1 - X)^{-1} = (1, 1, 1, \dots)$

Ring on Streams

Addition

$(a + b)_0 := a_0 + b_0$
$(a + b)' := a' + b'$

Subtraction

$(-a)_0 := -a_0$
$(-a)' := -a'$

Multiplication

$(a \times b)_0 := a_0 \times b_0$
$(a \times b)' := a_0 \times b' + a' \times b$

Division

$(s^{-1})_0 := s_0^{-1}$
$(s^{-1})' := -s' \times ([s_0] \times s)^{-1}$

Zero- and One-Elements

$0_0 := 0$
$0' := 0$
$1 := [1]$

Equalities on Streams

Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \wedge a' = b'$

Equalities on Streams

Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \land a' = b'$

Equality up-to

$a =_{n+1} b :\leftrightarrow a_0 = b_0 \land a' =_n b'$
$\forall a, b : a =_0 b$

Equalities on Streams

Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \land a' = b'$

Equality up-to

$a =_{n+1} b :\leftrightarrow a_0 = b_0 \land a' =_n b'$
$\forall a, b : a =_0 b$

$$(\forall n : a =_n b) \leftrightarrow a = b$$

Fibonacci revised

Fibonacci as Sequence

$F_0 := 1$
$F_1 := 1$
$F_{n+2} := F_n + F_{n+1}$

Fibonacci revised

Fibonacci as Sequence

$F_0 := 1$
$F_1 := 1$
$F_{n+2} := F_n + F_{n+1}$

Fibonacci as Stream

$fib_0 := 1$
$fib_1 := 1$
$fib'' := fib + fib'$

Fibonacci revised

Fibonacci as Sequence

$F_0 := 1$

$F_1 := 1$

$F_{n+2} := F_n + F_{n+1}$

Fibonacci as Stream

$fib_0 := 1$

$fib_1 := 1$

$fib'' := fib + fib'$

$$
\begin{aligned}
fib &= fib_0 + X \times fib' \\
&= fib_0 + X \times (fib_1 + X \times fib'') \\
&= 1 + X \times 1 + X^2 \times fib'' \\
&= 1 + X \times 1 + X^2 \times (fib + fib') \\
&= 1 + X \times (1 + X \times fib') + X^2 \times fib \\
&= 1 + X \times fib + X^2 \times fib
\end{aligned}
$$

Fibonacci revised

Fibonacci as Sequence

$$F_0 := 1$$
$$F_1 := 1$$
$$F_{n+2} := F_n + F_{n+1}$$

Fibonacci as Stream

$$fib_0 := 1$$
$$fib_1 := 1$$
$$fib'' := fib + fib'$$

$$\begin{aligned} fib &= fib_0 + X \times fib' \\ &= fib_0 + X \times (fib_1 + X \times fib'') \\ &= 1 + X \times 1 + X^2 \times fib'' \\ &= 1 + X \times 1 + X^2 \times (fib + fib') \\ &= 1 + X \times (1 + X \times fib') + X^2 \times fib \\ &= 1 + X \times fib + X^2 \times fib \end{aligned}$$

Fibonacci closed

$$fib = \frac{1}{1 - X - X^2}$$

Example of reasoning with Streams

Fibonacci closed

$$fib = \frac{1}{1-X-X^2}$$

$$fib' = \frac{1+X}{1-X-X^2}$$

Example of reasoning with Streams

Fibonacci closed

$fib = \frac{1}{1-X-X^2}$

$fib' = \frac{1+X}{1-X-X^2}$

Sum of Fibonacci

$\sum_{k=0}^{n} F_k$

$\frac{1}{1-X} \times \frac{1}{1-X-X^2}$

Example of reasoning with Streams

Fibonacci closed

$fib = \frac{1}{1-X-X^2}$

$fib' = \frac{1+X}{1-X-X^2}$

Sum of Fibonacci

$\sum_{k=0}^{n} F_k$

$\frac{1}{1-X} \times \frac{1}{1-X-X^2} = \frac{A}{1-X} + \frac{B}{1-X-X^2}$

Example of reasoning with Streams

Fibonacci closed

$fib = \frac{1}{1-X-X^2}$

$fib' = \frac{1+X}{1-X-X^2}$

Sum of Fibonacci

$\sum_{k=0}^{n} F_k$

$\frac{1}{1-X} \times \frac{1}{1-X-X^2} = \frac{-1}{1-X} + \frac{2+X}{1-X-X^2}$

Example of reasoning with Streams

Fibonacci closed

$fib = \frac{1}{1-X-X^2}$

$fib' = \frac{1+X}{1-X-X^2}$

Sum of Fibonacci

$\sum_{k=0}^{n} F_k$

$\frac{1}{1-X} \times \frac{1}{1-X-X^2} = \frac{-1}{1-X} + \frac{2+X}{1-X-X^2}$

$= -\frac{1}{1-X} + fib''$

Example of reasoning with Streams

Fibonacci closed

$fib = \frac{1}{1-X-X^2}$

$fib' = \frac{1+X}{1-X-X^2}$

Sum of Fibonacci

$\sum_{k=0}^{n} F_k = F_{n+2} - 1$

$\frac{1}{1-X} \times \frac{1}{1-X-X^2} = \frac{-1}{1-X} + \frac{2+X}{1-X-X^2}$

$= -\frac{1}{1-X} + fib''$

Implementing Streams in Coq

$$
\begin{aligned}
&\mathit{Stream}\ T := \mathbb{N} \to T \\
&\mathit{head}\ s := (s\ 0) \\
&\mathit{tail}\ s := \lambda n, s(n+1)
\end{aligned}
$$

Implementing Multiplication

Definition

$(a \times b)_0 := a_0 \times b_0$
$(a \times b)' := a_0 \times b' + a' \times b$

Implementation

$$
\begin{aligned}
a \times b := \lambda n, &\text{match } n \text{ with} \\
&|0 \Rightarrow a\,0 \times b\,0 \\
&|Sn' \Rightarrow (a_0 \times b' + a' \times b)\,n'
\end{aligned}
$$

Implementing Multiplication

Definition

$$(a \times b)_0 := a_0 \times b_0$$
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Implementation

$$\begin{aligned} a \times b := \lambda n, &\text{match } n \text{ with} \\ &|0 \Rightarrow a\,0 \times b\,0 \\ &|Sn' \Rightarrow (a_0 \times b')\,n' + (a' \times b)\,n' \end{aligned}$$

Implementing Division

Definition

$(s^{-1})_0 := s_0^{-1}$
$(s^{-1})' := -s' \times ([s_0] \times s)^{-1}$

Implementation

$$
\begin{aligned}
inv\ a := \lambda n, \text{match } n \text{ with} \\
&|0 \Rightarrow inv(a\,0) \\
&|Sn' \Rightarrow (-a' \times inv([a_0] \times a))\,n'
\end{aligned}
$$

Approximization of s^{-1}

tail: $\underbrace{Stream}_{\text{input}} \to Stream$

$$\begin{aligned} inv\ a := \lambda n, \text{match } n \text{ with} \\ &|0 \Rightarrow inv(a\,0) \\ &|Sn' \Rightarrow (-a' \times inv([a_0] \times a))\, n' \end{aligned}$$

pre-tail: $\underbrace{Stream}_{\text{input}} \to \underbrace{(Stream \to Stream)}_{\text{inverse}} \to Stream$

Approximization of s^{-1}

tail: $\underbrace{Stream}_{\text{input}} \to Stream$

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\begin{aligned}
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&|0 \Rightarrow inv(a\,0) \\
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\end{aligned}
$$

pre-tail: $\underbrace{Stream}_{\text{input}} \to \underbrace{(Stream \to Stream)}_{\text{inverse}} \to Stream$

$(pre\_inv\_tail\ s := \lambda inv, -s' \times inv([s_0] \times s))$

Approximization of s^{-1}

tail: $\underbrace{Stream}_{\text{input}} \to Stream$

$$\begin{aligned} inv\ a := \lambda n, &\text{match } n \text{ with} \\ &|0 \Rightarrow inv(a\,0) \\ &|Sn' \Rightarrow (-a' \times inv([a_0] \times a))\,n' \end{aligned}$$

pre-tail: $\underbrace{Stream}_{\text{input}} \to \underbrace{(Stream \to Stream)}_{\text{approximization}} \to Stream$

$(pre\_inv\_tail\ s := \lambda inv, -s' \times inv([s_0] \times s))$

Approximization of s^{-1}

tail: $\underbrace{Stream}_{\text{input}} \to Stream$

$$
\begin{aligned}
inv\, a := \lambda n, \text{match } n \text{ with} \\
&|0 \Rightarrow inv(a\,0) \\
&|Sn' \Rightarrow (-a' \times inv([a_0] \times a))\, n'
\end{aligned}
$$

pre-tail: $\underbrace{Stream}_{\text{input}} \to \underbrace{(Stream \to Stream)}_{\text{approximization}} \to Stream$

$(pre\_inv\_tail\, s := \lambda inv, -s' \times inv([s_0] \times s)$

Causality

$causal\, \phi :\Leftrightarrow \forall n, a1, a2 : (\forall s : a1\, s =_n a2\, s) \to \phi\, a1 =_n \phi\, a2$

Approximization of s^{-1}

tail: $\underbrace{Stream}_{\text{input}} \to Stream$

$$
\begin{aligned}
inv\, a := \lambda n, &\text{match } n \text{ with} \\
&|0 \Rightarrow inv(a\,0) \\
&|Sn' \Rightarrow (-a' \times inv([a_0] \times a))\, n'
\end{aligned}
$$

pre-tail: $\underbrace{Stream}_{\text{input}} \to \underbrace{(Stream \to Stream)}_{\text{approximization}} \to Stream$

$(pre\_inv\_tail\, s := \lambda inv, -s' \times inv([s_0] \times s)$

Causality

$$causal\, \phi :\Leftrightarrow \forall n, a1, a2 : (\forall s : a1\, s =_n a2\, s) \to \phi\, a1 =_n \phi\, a2$$

$$C\, Stream := \{\phi : (Stream \to Stream) \to Stream | causal\, \phi\}$$

Approximization of s^{-1}

tail: $\underbrace{Stream}_{\text{input}} \rightarrow Stream$

$$\begin{aligned} inv\, a := \lambda n, &\text{match } n \text{ with} \\ &|0 \Rightarrow inv(a\, 0) \\ &|Sn' \Rightarrow (-a' \times inv([a_0] \times a))\, n' \end{aligned}$$

pre-tail: $\underbrace{Stream}_{\text{input}} \rightarrow C\; Stream$

$(pre\_inv\_tail\; s := \lambda inv, -s' \times inv([s_0] \times s)$

Causality

$causal\, \phi :\Leftrightarrow \forall n, a1, a2 : (\forall s : a1\, s =_n a2\, s) \rightarrow \phi\, a1 =_n \phi\, a2$

$C\; Stream := \{\phi : (Stream \rightarrow Stream) \rightarrow Stream | causal\, \phi\}$

By this: stepwise approximazation of s^{-1}.

Implementing Division (cnt.)

Definition

$(s^{-1})_0 := s_0^{-1}$
$(s^{-1})' := -s' \times ([s_0] \times s)^{-1}$

Implementation

$$\begin{aligned} &inv\_head\ a := inv(a\ 0) \\ &pre\_inv\_tail := \lambda inv\_app, -a' \times inv\_app([a_0] \times a) \\ &inv := approx\_stream\ inv\_head\ pre\_inv\_tail \end{aligned}$$

Approximazation revised

Causality

$$causal\,\phi :\Leftrightarrow \forall n, a1, a2 : (\forall s : a1\,s =_n a2\,s) \rightarrow \phi\,a1 =_n \phi\,a2$$

$$C\;Stream := \{\phi : (Stream \rightarrow Stream) \rightarrow Stream | causal\,\phi\}$$

pre-tail: $\underbrace{Stream}_{\text{input}} \rightarrow C\;Stream$

Approximazation revised

Causality

$$causal\,\phi :\Leftrightarrow \forall n, a1, a2 : (\forall s : a1\,s =_n a2\,s) \rightarrow \phi\,a1 =_n \phi\,a2$$

$$C\,X := \{\phi : (X \rightarrow Stream) \rightarrow Stream | causal\,\phi\}$$

pre-tail: $\underbrace{X}_{\text{input}} \rightarrow C\,X$

Approximazation revised

Causality

$$causal\, \phi :\Leftrightarrow \forall n, a1, a2 : (\forall s : a1\, s =_n a2\, s) \rightarrow \phi\, a1 =_n \phi\, a2$$

$$C\, X := \{\phi : (X \rightarrow Stream) \rightarrow Stream | causal\, \phi\}$$

pre-tail: $\underbrace{X}_{\text{input}} \rightarrow C\, X$

Way to solve a system of causal differential equations by approximazation.

Summary

- Streams as simple and clean representation of sequences
- A way to workaround Coq's restrictions with respect to coinductive definitions.
- A way to solve a system of causal differential equations

Further Reading

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Implementing Equalities

Definition Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \wedge a' = b'$

Implementing Equalities

Definition Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \wedge a' = b'$

Implementation

$a = b := \forall n, a\, n = b\, n$

Implementing Equalities

Definition Equality up-to

$a =_{n+1} b :\leftrightarrow a_0 = b_0 \wedge a' =_n b'$

$\forall a, b : a =_0 b$

Definition Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \wedge a' = b'$

Implementation

$a = b := \forall n, a\, n = b\, n$

Implementing Equalities

Definition Pointwise Equality

$a = b :\leftrightarrow a_0 = b_0 \land a' = b'$

Implementation

$a = b := \forall n, a\, n = b\, n$

Definition Equality up-to

$a =_{n+1} b :\leftrightarrow a_0 = b_0 \land a' =_n b'$
$\forall a, b : a =_0 b$

Implementation

$$\frac{}{r =_0 s}$$

$$\frac{r_0 = s_0 \qquad r' =_n s'}{r =_{n+1} s}$$

Implementing Addition

Definition

$$(a+b)_0 := a_0 + b_0$$
$$(a+b)' := a' + b'$$

Implementation

$$\begin{aligned} a+b := \lambda n, &\text{match } n \text{ with} \\ &|0 \Rightarrow a\,0 + b\,0 \\ &|Sn' \Rightarrow (a'+b')n' \end{aligned}$$

Implementing Addition

Definition

$(a + b)_0 := a_0 + b_0$
$(a + b)' := a' + b'$

Implementation

$$a + b := \lambda n, a\,n + b\,n$$