Towards a Formal Proof of the Cook-Levin Theorem

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Final Bachelor Talk

The Cook-Levin Theorem

SAT is NP-complete [Cook, 1971, Levin, 1973].

SAT

Given a Boolean formula in conjunctive normal form, does there exist a satisfying assignment?

[Karp, 1972]: 21 more combinatorial NP-complete problems

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Mechanisation [Gamboa and Cowles, 2004]: verified construction without connection to computational model (ACL2)

Our Computational Model

Prevalent computational model: Turing machines

> *Turing machines as model of computation are inherently infeasible for the formalisation of any computability- or complexity-theoretic result*
>
> [Forster et al., 2020]

Here: use λ-calculus L
Supported by:

- reasonability of L [Forster et al., 2019]
- extraction mechanism in Coq [Forster and Kunze, 2019]

size explosion: **"time bounds space" does not hold**

Basic Complexity Theory

Problems $P, Q : X \to \mathbb{P}$
NP: verifier characterisation

$$P \preceq_p Q := \exists f : X \to Y, \quad (\forall x, P\ x \leftrightarrow Q(f\ x))$$
$$\wedge \text{ polynomial-time computable } f$$
$$\wedge\ f \text{ increases the size polynomially}$$

$$\text{NP-hard } P := \forall Q, Q \in \text{NP} \to Q \preceq_p P$$
$$\text{NP-complete } P := P \in \text{NP} \wedge \text{NP-hard } P$$

Proving the Cook-Levin Theorem

SAT is NP-complete.

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Given a Boolean formula in conjunctive normal form, does there exist a satisfying assignment?

Idea: encode computation as Boolean formula

L: non-local computations, too high-level ☹

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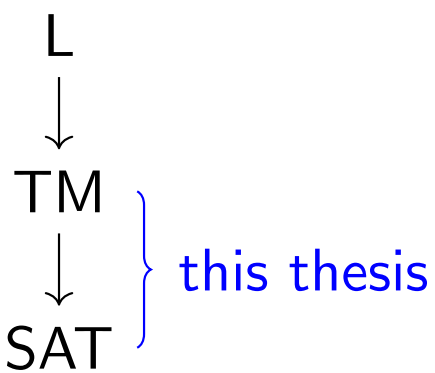

Generic Problem for Turing Machines

- formalisation by [Asperti and Ricciotti, 2015] and mechanisation from [Forster et al., 2020]
- two-sided infinite tapes
- no explicit blanks

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TMGenNP

$$\textbf{TMGenNP}\ (M, \mathit{inp}, t) := M \text{ is a } \mathit{nondet.} \text{ 1-tape TM}$$
$$\wedge M \text{ accepts on } \mathit{inp} \text{ in } \leq t \text{ steps}$$

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TMGenNP

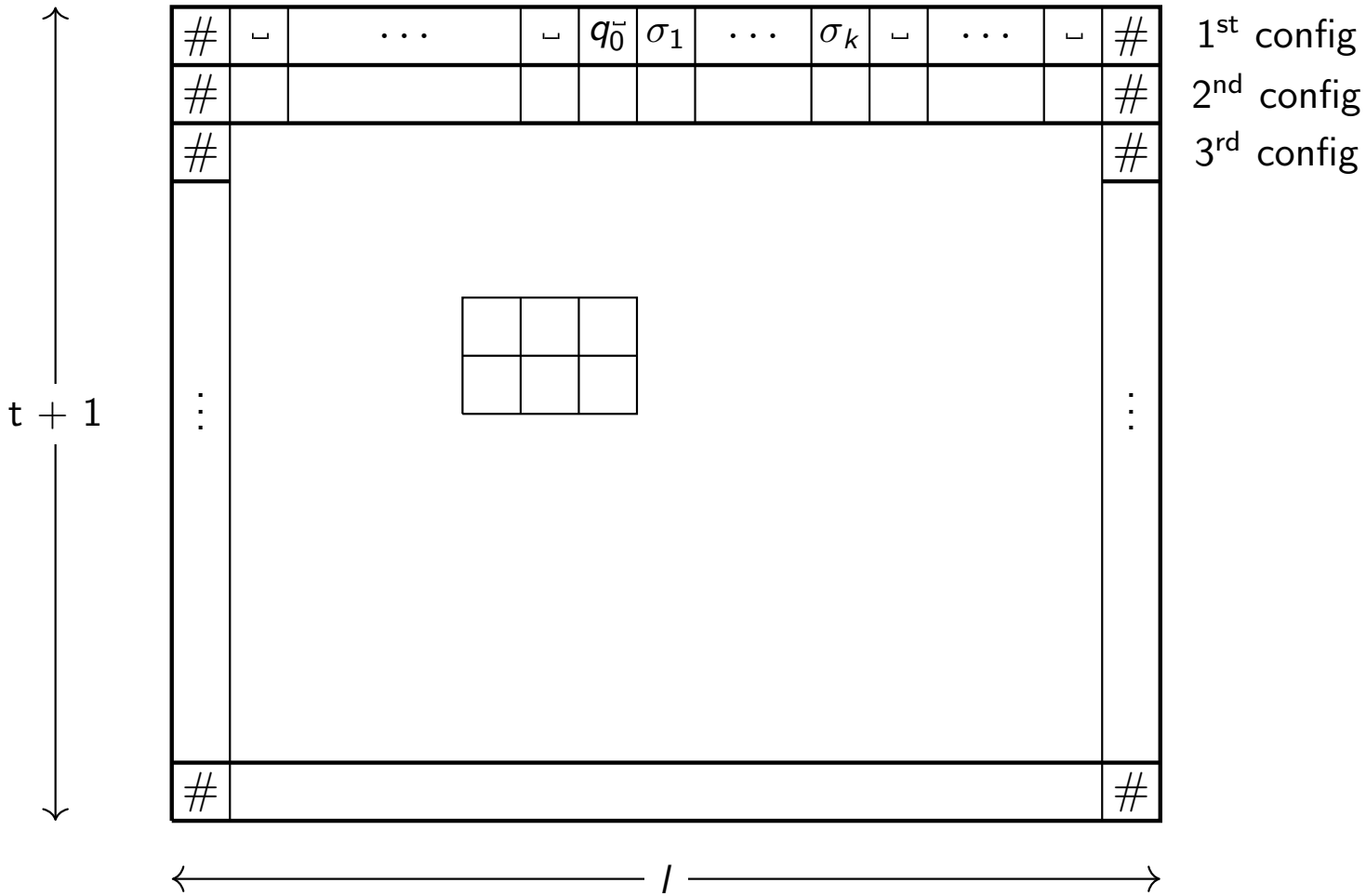
$$\begin{aligned}\textbf{TMGenNP}\ (M, \mathit{inp}, k', t) := {} & M \text{ is a } \mathit{det.} \text{ 1-tape TM} \\ \wedge & \exists\ \mathit{cert}, |\mathit{cert}| \leq k' \\ \wedge & M \text{ accepts on } \mathit{inp} +\!\!+ \mathit{cert} \text{ in } \leq t \text{ steps}\end{aligned}$$

Bounded Size

SAT formula has a fixed size, but:

- TM may have different space usage depending on input
- TM may take a different number of steps until it halts

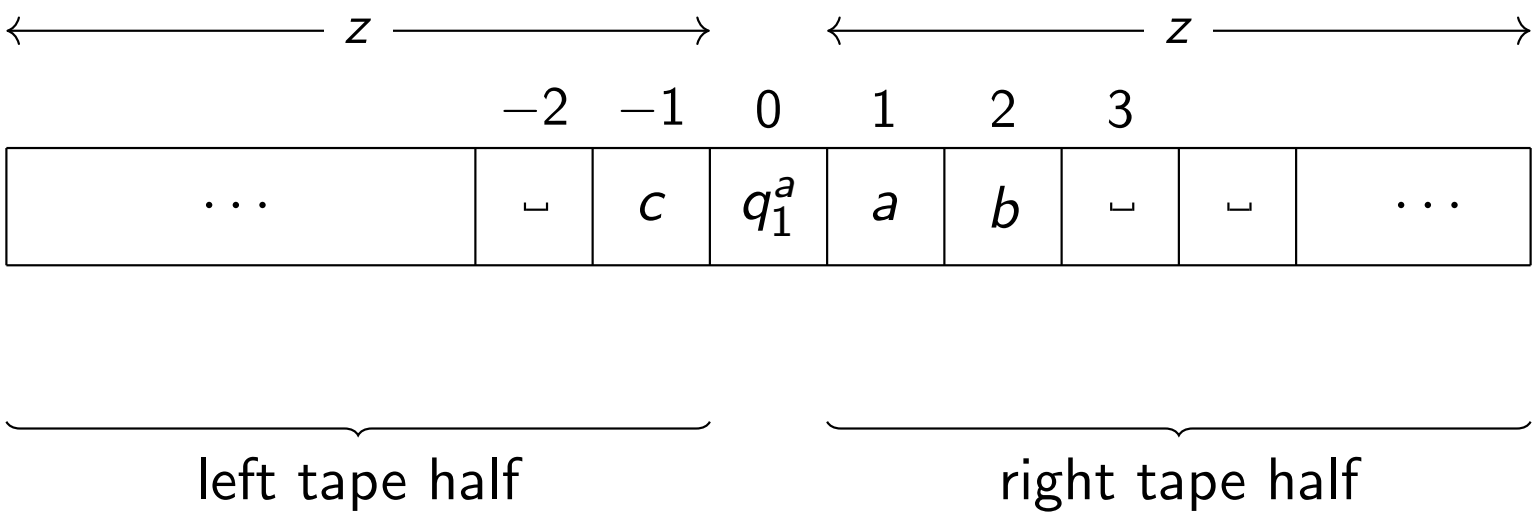
Tableau of Configurations[1]



[1] based on [Sipser, 1997], similar to [Cook, 1971]

Configuration String

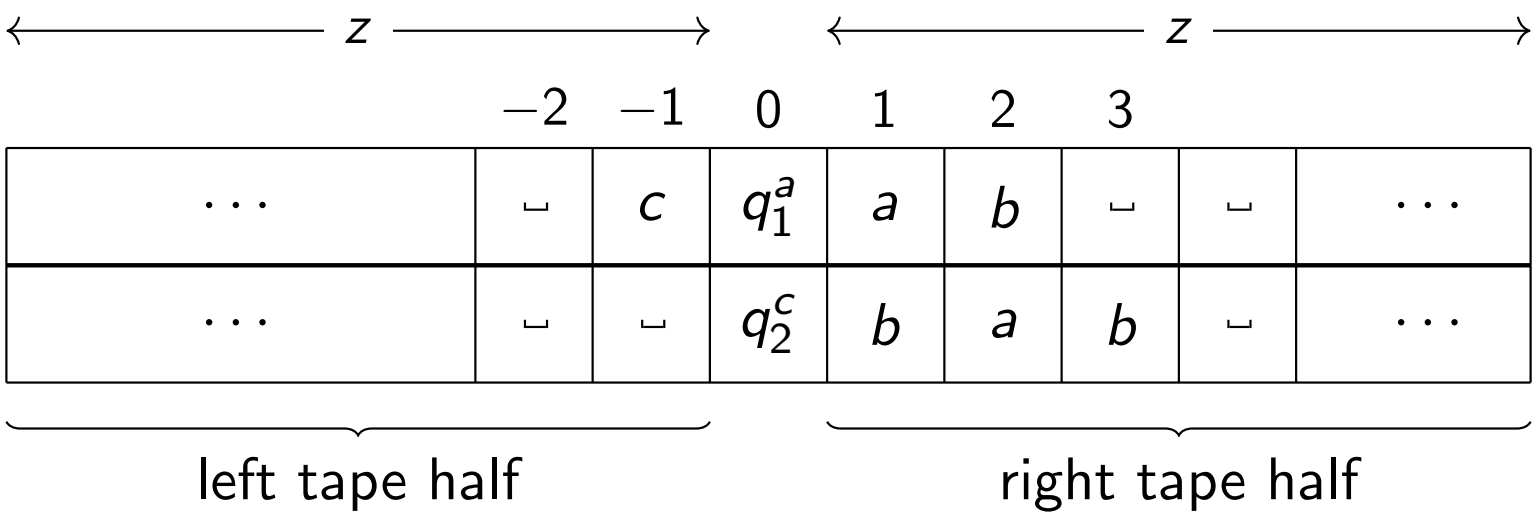
$\Sigma_{\mathsf{TM}} = \{a, b, c\}$



special blanks ␣ for unused regions of the string

Configuration String

$\Sigma_{\mathsf{TM}} = \{a, b, c\}$ $\qquad \delta(q_1, a) = (q_2, {}^{\circ}b, \mathsf{L})$



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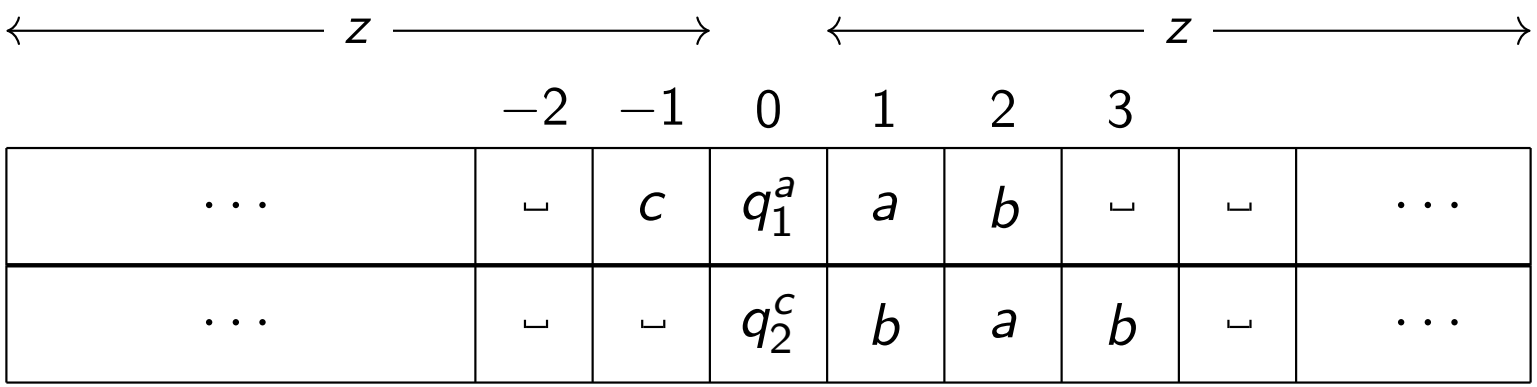
Rewrite Windows: Force Valid Configuration Changes

$\Sigma_{\mathsf{TM}} = \{a, b, c\}$ $\qquad$ $\delta(q_1, a) = (q_2, {}^{\circ}b, \mathsf{L})$



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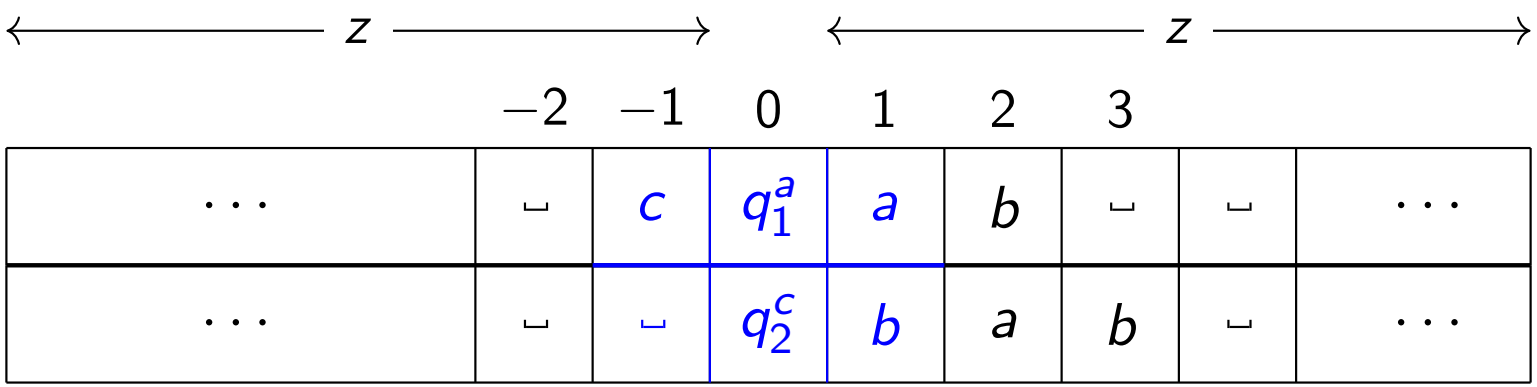

c	q_1^a	a
␣	q_2^c	b

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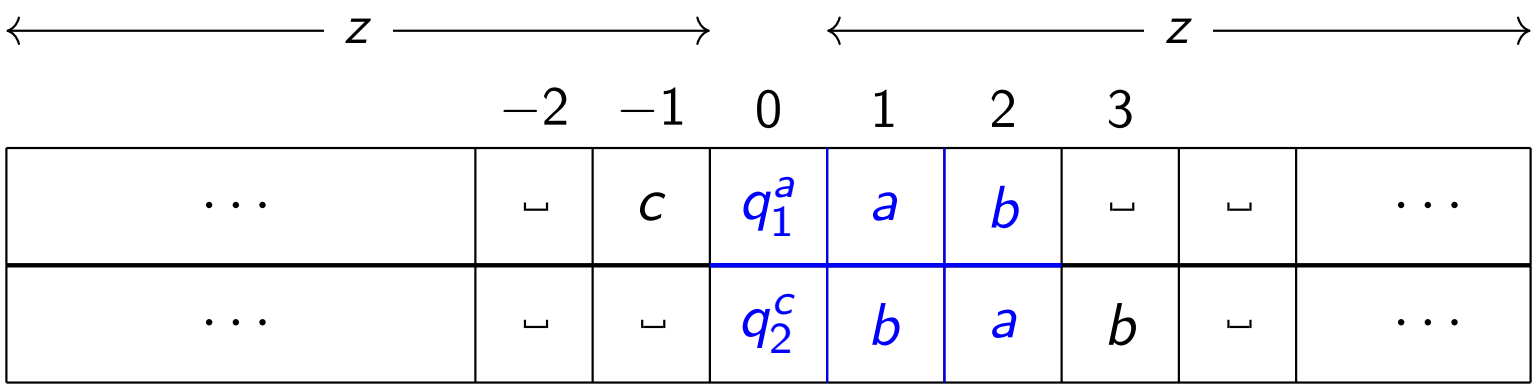

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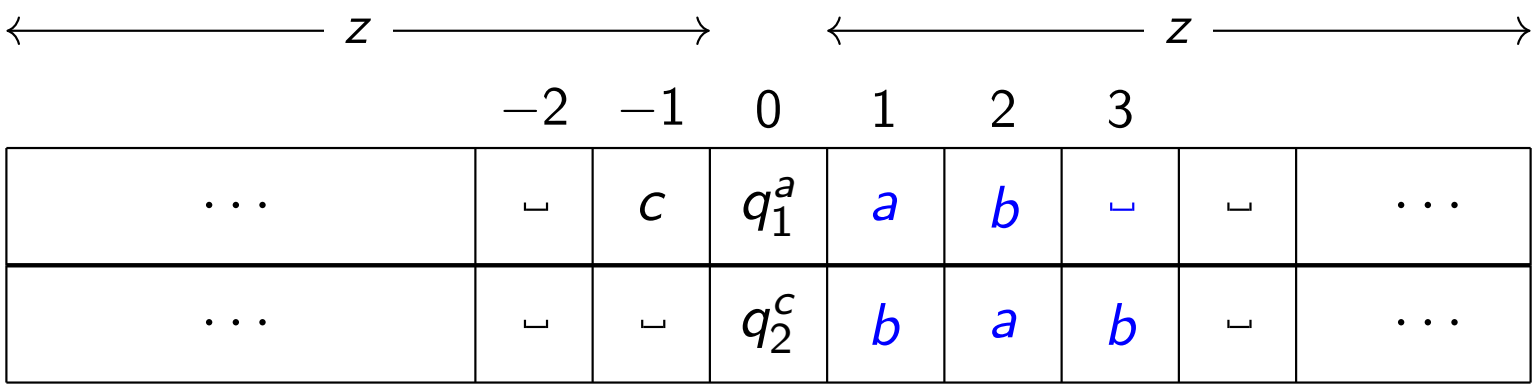

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q_1^a	a	b
q_2^c	b	a

a	b	␣
b	a	b

Parallel Rewriting (**PR**)

Given:

- an initial string $x_0 \in \Sigma^l$ over an alphabet Σ
  representation of initial config
- a set of rewrite windows R of width w
  possible local behaviours of the Turing machine
- a step count t
  number of TM steps
- a set of final substring constraints R_{final}
  symbols of accepting states

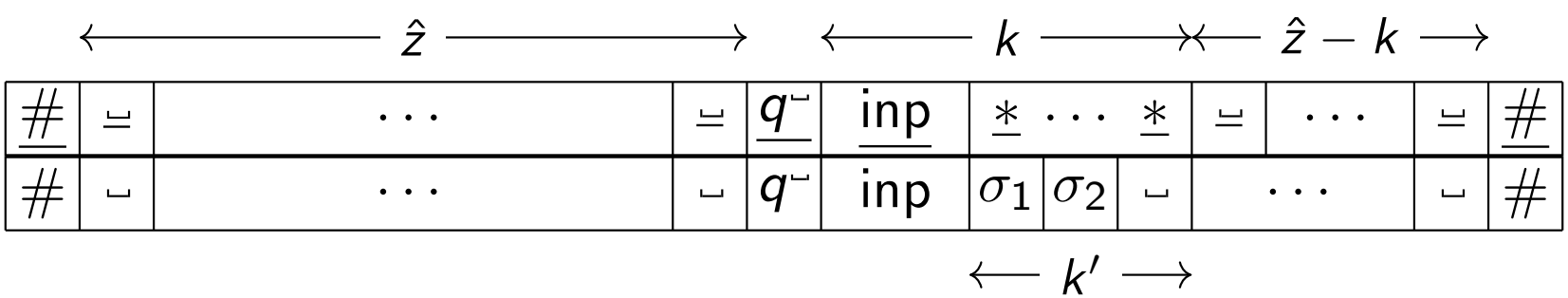
Determine: $\exists\, x_1, \ldots, x_t \in \Sigma^l$ s.t.

- $x_i \rightsquigarrow x_{i+1}$: "for all offsets there exists a rewrite window"
- there exists an element $x \in R_{final}$ which is a substring of x_t

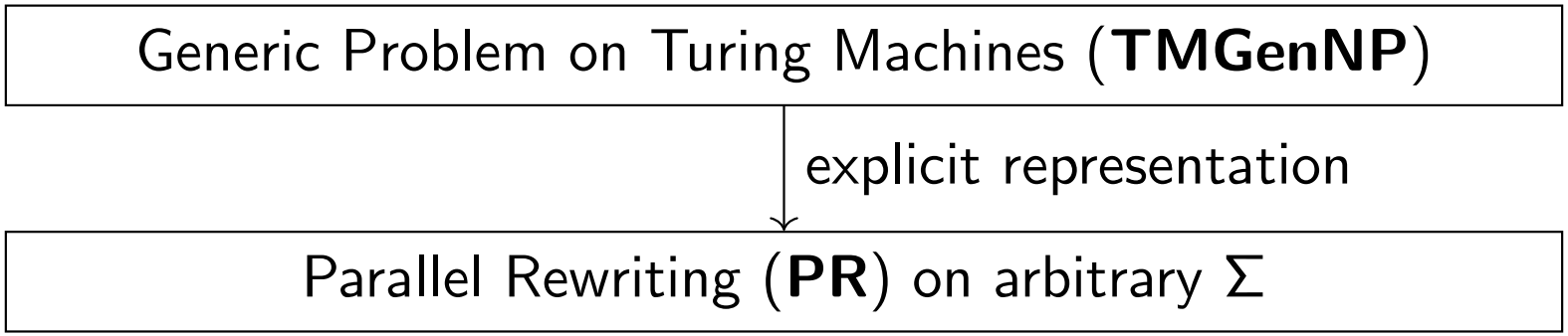
Nondeterministic Certificate

- "guess" certificate of length $\leq k'$ with a single rewrite step
- add symbols $\{\underline{\#}, \underline{\sqcup}, \underline{*}, \underline{q^{\sqcup}}, \underline{\sigma}\}$ for initial state q and $\sigma : \Sigma_{\mathsf{TM}}$

Initial string:



Chain of Reductions



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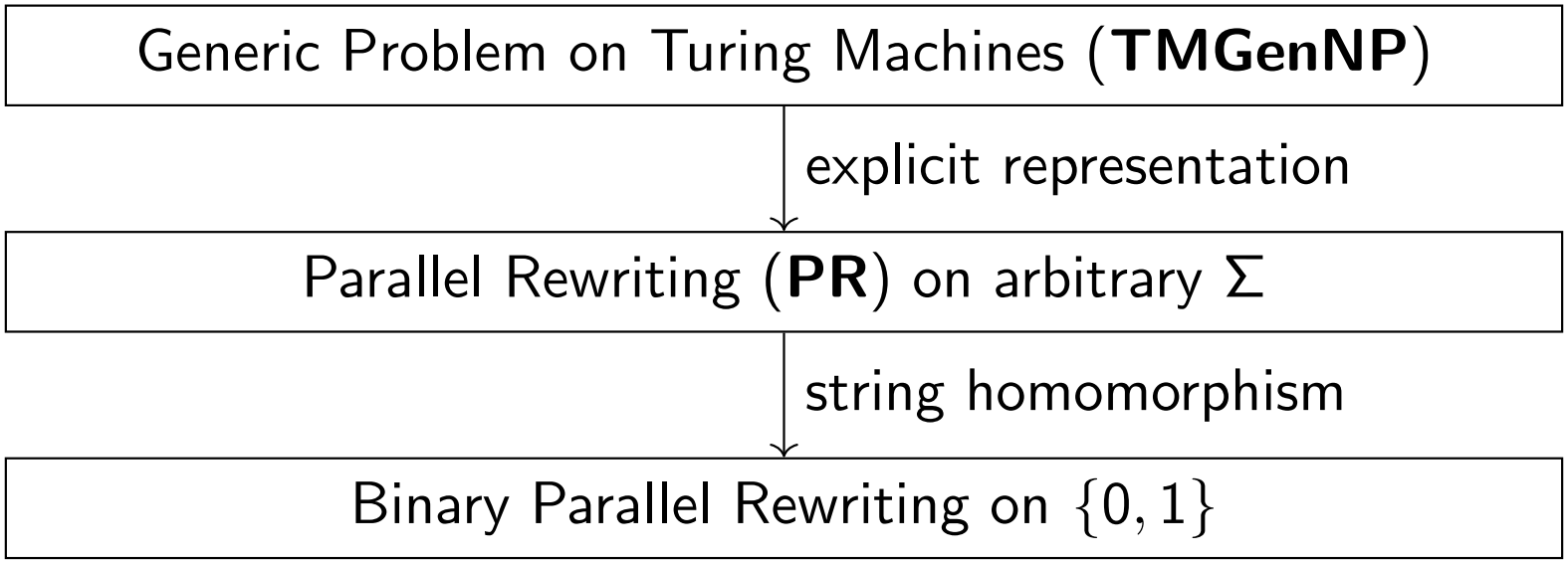

To a Binary Alphabet

- arbitrary alphabet $\Sigma = \{\sigma_0, \ldots, \sigma_{|\Sigma|-1}\}$
- string homomorphism $h : \Sigma^* \to \{0,1\}^*$

$$\sigma_i \mapsto 0^i 10^{|\Sigma|-i-1}$$

- in general: injective *uniform* homomorphisms
  $\to$ scale length up by constant factor

Example ($\Sigma = \{a, b\}$):

$$a \mapsto 01$$
$$b \mapsto 10$$

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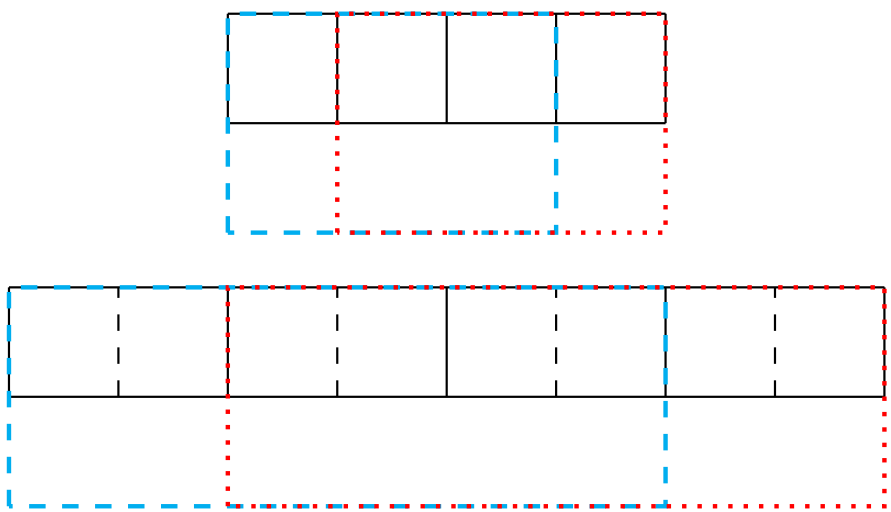

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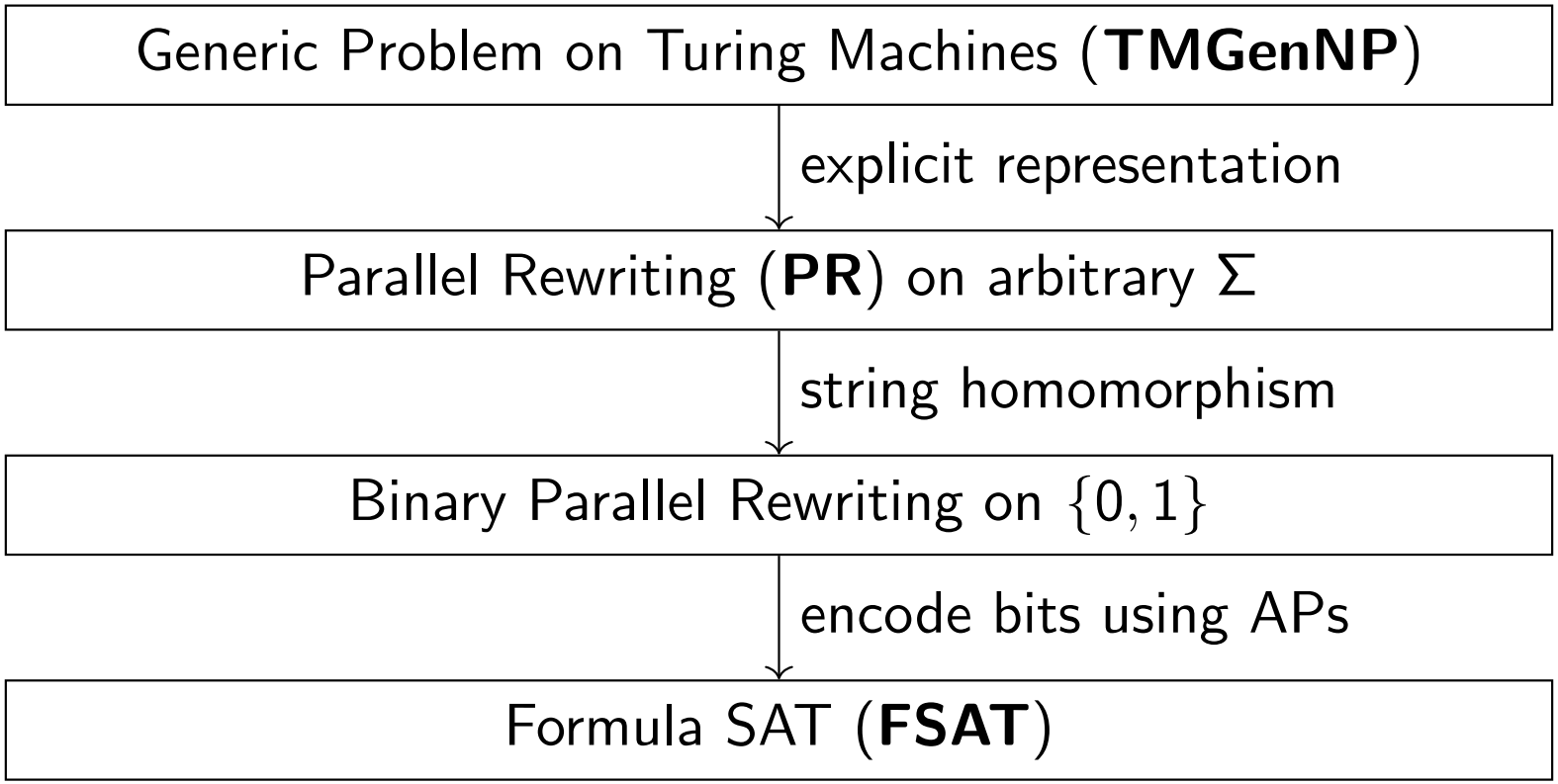
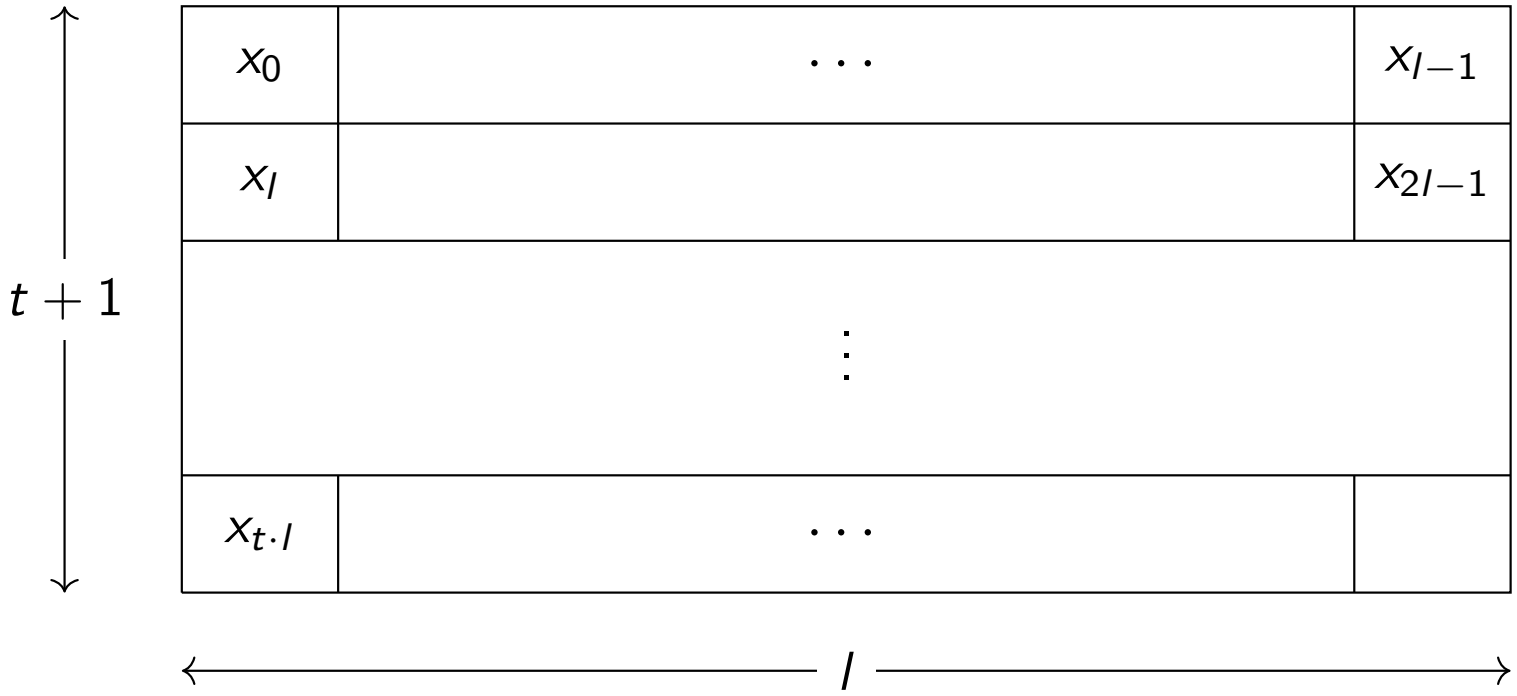


Encoding as a Boolean Formula

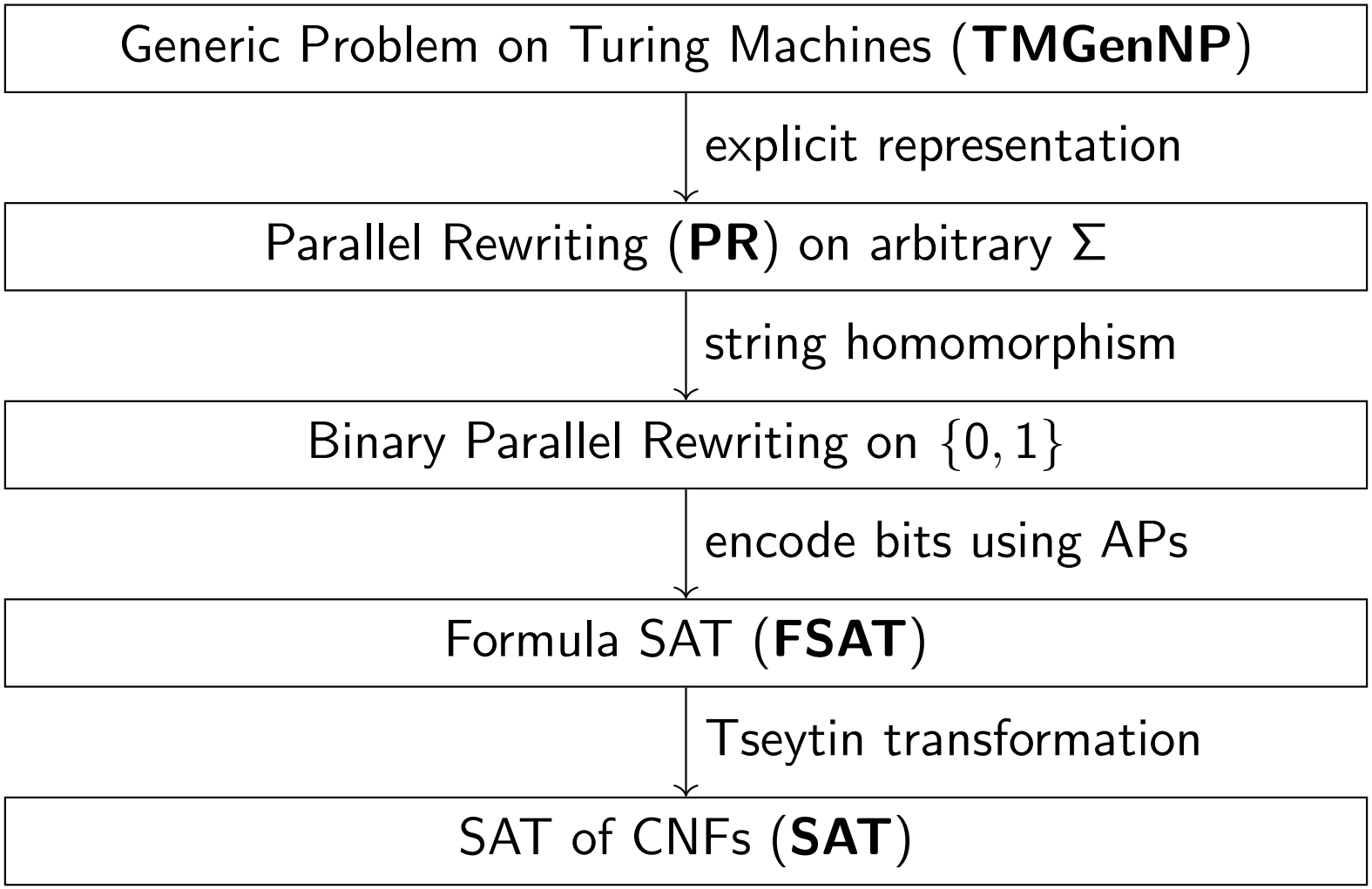


$$\Phi := \Phi_{\text{init}} \wedge \Phi_{\text{rewrite}} \wedge \Phi_{\text{final}}$$

$$\Phi_{\text{rewrite}} := \bigwedge_{\text{rewrites}} \bigwedge_{\text{offsets}} \bigvee_{w \in R} \bigwedge_{\text{characters of } w} \text{encodeBit}$$

$x_i \rightsquigarrow x_{i+1}$: "for all offsets there exists a rewrite window"

Chain of Reductions



Contributions

Main result

If **TMGenNP** is NP-hard, then **SAT** is NP-complete.

- factorisation of the textbook proof [Sipser, 1997] to make mechanisation feasible: new **PR** problem
- full running-time analysis in L

Moreover:

- **TMGenNP** $\in$ NP
- reduction k-**SAT** $\preceq_p$ **Clique**
- conditional NP-completeness of **Clique**
- formalisation of $\#P$

Future Work

- missing reduction from L to **TMGenNP**
- space-related results, e.g. Savitch
- binary encodings

LOC

Component		Spec	Proof
complexity definitions by Fabian		290	506
preliminaries		281	634
flat finite types		83	223
problem definitions + NP proofs		1093	1860
TMGenNP to **PR**	correctness	1843	2481
	time analysis	838	1706
3-PR to **PR**		37	174
PR to **BinaryPR**		222	719
BinaryPR to **FSAT**		312	1078
FSAT to **SAT**		213	605
k-**SAT** to **Clique**		386	773
total		5230	10038
without extraction		3594	6857
without computability		2339	4524

Comparison with [Gamboa and Cowles, 2004]

- existing formalisation in ACL2
- differences in setting:
  - single-sided infinite tapes
  - explicit nondeterminism instead of verifiers
  - reduction only to **FSAT**
- *direct* reduction, no factorisation $\rightarrow$ only seems to be feasible because of restricted setting
- no running time analysis with respect to reasonable model

Rewrite Rules

Add one symbol to the right half of the tape:

a	b	$\sqcup$	$\sqcup$	$\cdots$
b	a	b	$\sqcup$	$\cdots$

a	b	$\sqcup$
b	a	b

Rewrite Rules

Add one symbol to the right half of the tape:

σ_1	σ_2	$\sqcup$	$\sqcup$	$\cdots$
σ_3	σ_1	σ_2	$\sqcup$	$\cdots$

$$\sigma_i \in \Sigma_{\mathsf{TM}}$$

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$\sqcup$	$\sqcup$	$\sqcup$
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$\cdots$

Tape Shifts

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σ_1	σ_2	$\sqcup$
σ_1	σ_2	$\sqcup$

$\cdots$	$\sqcup$	c	q_1^a	b	b	$\sqcup$	$\sqcup$	$\cdots$
$\cdots$	$\sqcup$	$\sqcup$	q_2^c	b	b	b	$\sqcup$	$\cdots$

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Polarities $\{\overleftarrow{\cdot}, \overline{\cdot}, \overrightarrow{\cdot}\}$

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Transition Rules – Example

$m \in \Sigma_{\mathsf{TM}} \cup \{\sqcup\}, \sigma \in \Sigma_{\mathsf{TM}}$

$\delta(q, a) = (p, {}^{\circ}b, L)$:

$\sqcup$	q^a	m_1
$\sqcup$	$p^{\sqcup}$	$\overrightarrow{b}$

σ_1	q^a	m_1
$\overrightarrow{m_2}$	p^{σ_1}	$\overrightarrow{b}$

$\sqcup$	$\sqcup$	q^a
$\sqcup$	$\sqcup$	$p^{\sqcup}$

$\sqcup$	σ_1	q^a
$\sqcup$	$\sqcup$	p^{σ_1}

σ_1	σ_2	q^a
$\overrightarrow{m_1}$	$\overrightarrow{\sigma_1}$	p^{σ_2}

q^a	$\sqcup$	$\sqcup$
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In Coq mechanisation:
$\delta(q, a) = (p, {}^{\circ}b, L)$:

m_1	q^a	m_2
$\overrightarrow{m_3}$	p^{m_1}	$\overrightarrow{b}$

m_1	m_2	q^a
$\overrightarrow{m_3}$	$\overrightarrow{m_1}$	p^{m_2}

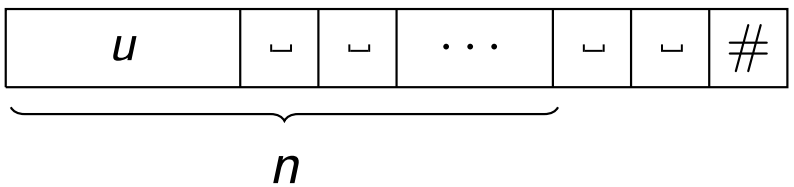
q^a	m_1	m_2
p^{m_3}	$\overrightarrow{b}$	$\overrightarrow{m_1}$

Induces garbage, e.g.

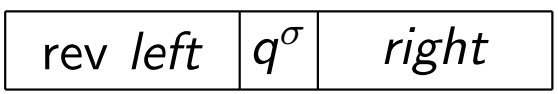
$\sqcup$	q^a	$\sqcup$
$\overrightarrow{\sigma}$	$p^{\sqcup}$	$\overrightarrow{b}$

Representation Relations

Representation of tape halves:

$u \sim_t^n$ 

Representation of configurations:

$q; (ls, \sigma, rs) \sim_c$ 

, where:

- $ls \sim_t^{z'} left$
- $rs \sim_t^{z'} right$

Deterministic Simulation

$$
\begin{array}{rcl}
q;(ls,\sigma,rs) & \sim_c & \boxed{\text{rev } left \mid q^{\sigma} \mid right} \\
\Upsilon & & \wr \\
q';(ls',\sigma',rs') & \sim_c & \boxed{\text{rev } left' \mid q'^{\sigma'} \mid right'}
\end{array}
$$

where $ls \sim_t^{z'} left, rs \sim_t^{z'} right$ and $ls' \sim_t^{z'} left', rs' \sim_t^{z'} right'$

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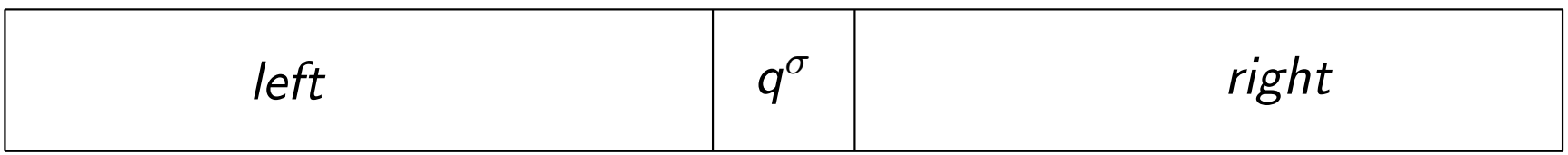

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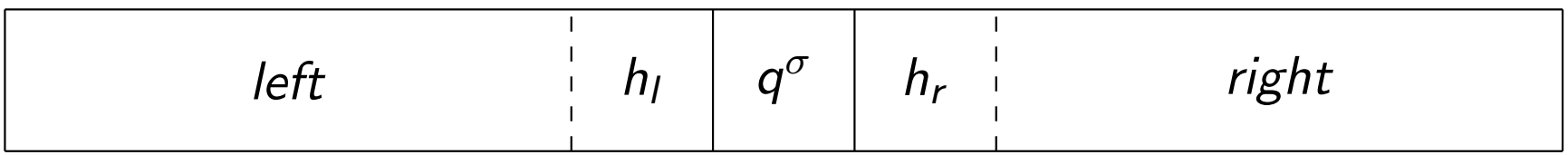
$$

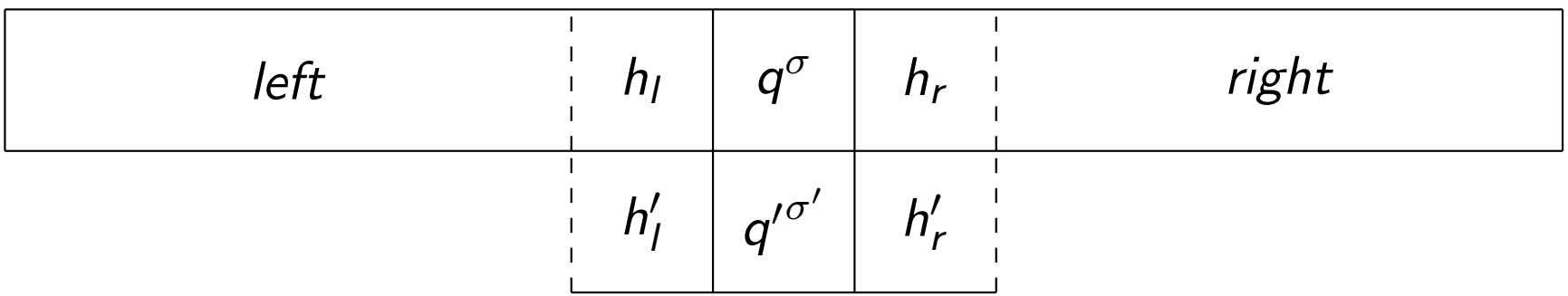
where $ls \sim_t^{z'} left, rs \sim_t^{z'} right$ and $ls' \sim_t^{z'} left', rs' \sim_t^{z'} right'$



Deterministic Simulation

$$q;(ls,\sigma,rs) \sim_c \boxed{\text{rev } left \mid q^{\sigma} \mid right}$$
$$\curlyvee \qquad\qquad \wr$$
$$q';(ls',\sigma',rs') \sim_c \boxed{\text{rev } left' \mid q'^{\sigma'} \mid right'}$$

where $ls \sim_t^{z'} left, rs \sim_t^{z'} right$ and $ls' \sim_t^{z'} left', rs' \sim_t^{z'} right'$



Deterministic Simulation

$$
\begin{array}{lll}
q;(ls,\sigma,rs) & \sim_c & \boxed{\text{rev } left \mid q^{\sigma} \mid right} \\
\Upsilon & & \wr \\
q';(ls',\sigma',rs') & \sim_c & \boxed{\text{rev } left' \mid q'^{\sigma'} \mid right'}
\end{array}
$$

where $ls \sim_t^{z'} left, rs \sim_t^{z'} right$ and $ls' \sim_t^{z'} left', rs' \sim_t^{z'} right'$

$left$	h_l	q^{σ}	h_r	$right$
$\exists! left'$	h'_l	$q'^{\sigma'}$	h'_r	$\exists! right'$

Tape Transformations

Add symbol:

$$rs \sim_t h \wedge |rs| < z' \rightarrow \exists! h', (h \rightsquigarrow \overrightarrow{a} :: h') \wedge a :: rs \sim_t^+ \overrightarrow{a} :: h'$$

$$ls \sim_t h \wedge |ls| < z' \rightarrow \exists! h', (\mathrm{rev} h \rightsquigarrow \mathrm{rev} \overleftarrow{a} :: h') \wedge a :: ls \sim_t^- \overleftarrow{a} :: h'$$

Remove symbol:

$$a :: b :: rs \sim_t a :: b :: h \rightarrow \exists! h', (a :: b :: h \rightsquigarrow \overleftarrow{b} :: h') \wedge b :: rs \sim_t^- \overleftarrow{b} :: h'$$

Leave unchanged:

$$a :: rs \sim_t a :: h \rightarrow \exists! h', (a :: h \rightsquigarrow \overline{a} :: h) \wedge a :: rs \sim_t^\circ \overline{a} :: h'$$

Main Simulation Results

Completeness

Let $(q, tape)$ be a configuration with $|tape| \leq k$. There exists s with $(q, tape) \sim_c s$.
If $(q, tape) \triangleright^{\leq t} (q', tape')$, then there exists s' with $s \rightsquigarrow^t s'$, $(q', tape') \sim_c s'$ and $s' \models R_{\text{final}}$.

Soundness

Let s be given such that $(q, tape) \sim_c s$ and $|tape| \leq k$ for some $q, tape$.
If $s \rightsquigarrow^t s'$ and $s' \models R_{\text{final}}$, then there exists $(q', tape')$ with $(q', tape') \sim_c s'$ such that $(q, tape) \triangleright^{\leq t} (q', tape')$ and $|tape'| \leq z'$.

Formulas: Representation of Predicates

$$f : \mathcal{F} := \mathsf{T} \mid v \mid f_1 \vee f_2 \mid f_1 \wedge f_2 \mid \neg f \qquad (v : \mathsf{var} := \mathbb{N})$$

- explicit assignments $e : \mathcal{L}(\mathbb{B})$ to variables $[s, s + |e|)$
- encoding of predicates $p : \mathcal{L}(\mathbb{B}) \to \mathbb{P}$:

  $\mathsf{encodesPredAt}\ \mathsf{start}\ l\ f\ p := \forall a, a \models f \leftrightarrow p\ (a[\mathsf{start}, \mathsf{start} + l])$

- $\mathsf{encodeBit}\ \mathsf{var}\ \mathsf{bit} :=$ **if** bit **then** var **else** $\neg$var

  $\mathsf{encodesPredAt}\ v\ 1\ (\mathsf{encodeBit}\ v\ b)(\lambda e.e = [s])$

The Tseytin Transformation

Goal: convert formula to CNF
Example: $f = f_1 \vee f_2$

Naive approach:

$$f = f_1 \vee f_2 \rightsquigarrow \quad N_1 \quad \vee \quad N_2$$

N : cnf $\quad$ C : clause $\quad$ v : var $\quad$ $f : \mathcal{F}$

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$$f = f_1 \vee f_2 \rightsquigarrow (C_1 \wedge C_2) \vee (C_3 \wedge C_4)$$

N : cnf $\quad$ C : clause $\quad$ v : var $\quad$ $f : \mathcal{F}$

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$$f = f_1 \vee f_2 \rightsquigarrow (C_1 \wedge C_2) \vee (C_3 \wedge C_4)$$
$$\leftrightarrow (C_1 \vee C_3) \wedge (C_1 \vee C_4) \wedge (C_2 \vee C_3) \wedge (C_2 \wedge C_4)$$

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$$\leftrightarrow (C_1 \vee C_3) \wedge (C_1 \vee C_4) \wedge (C_2 \vee C_3) \wedge (C_2 \wedge C_4)$$

exponential blowup!

N : cnf $\quad$ C : clause $\quad$ v : var $\quad$ $f : \mathcal{F}$

The Tseytin Transformation

Goal: convert formula to CNF
Example: $f = f_1 \vee f_2$

Solution: introduce new variables: $f \mapsto (v, N)$ s.t.

$$f \left|\!\!\models\!\!\right| (v, N) := \mathbf{SAT} f \leftrightarrow \mathbf{SAT}(v \wedge N)$$

N : cnf $\quad$ C : clause $\quad$ v : var $\quad$ $f : \mathcal{F}$

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$$f_1 \left|\!\models\!\right| (v_1, N_1) \qquad f_2 \left|\!\models\!\right| (v_2, N_2)$$

Goal: $f = f_1 \vee f_2 \left|\!\models\!\right| (v, N)$ for fresh v

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$$N := \underbrace{N_1}_{f_1 \leftrightarrow v_1} \wedge \underbrace{N_2}_{f_2 \leftrightarrow v_2} \wedge ?$$

N : cnf $\qquad$ C : clause $\qquad$ v : var $\qquad$ $f : \mathcal{F}$

The Tseytin Transformation

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Example: $f = f_1 \vee f_2$

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Goal: $f = f_1 \vee f_2 \models\!\!\models (v, N)$ for fresh v

$$N := \underbrace{N_1}_{f_1 \leftrightarrow v_1} \wedge \underbrace{N_2}_{f_2 \leftrightarrow v_2} \wedge (v \leftrightarrow (v_1 \vee v_2))$$

$$\Rightarrow f \leftrightarrow v$$

N : cnf $\qquad$ C : clause $\qquad$ v : var $\qquad$ $f : \mathcal{F}$

Tseytin: Correctness Relation

$$\text{tseytin}' : \underbrace{\text{var}}_{\text{next free var}} \to \mathcal{F} \to \underbrace{\text{var}}_{\text{representing variable}} \times \text{cnf} \times \underbrace{\text{var}}_{\text{next free var}}$$

$$f \left|\!\!\left|\!\!\frac{nf, nf'}{b}\right|\!\!\right| (v, N), \text{ iff}$$

- $N \subseteq ([0, b) \cup [nf, nf'))$,
- $v \in [nf, nf')$,
- for all $a \subseteq [0, b)$, there exists $a' \subseteq [nf, nf')$ such that $(a \cup a') \models N$,
- and for all a with $a \models N$, the equivalence $a \models v \leftrightarrow a \models f$ holds.

Basic Definitions: NP[2]

Problem $Q : X \to \mathbb{P}$

$Q \in \mathrm{NP}$ iff:

- there is a verifier $V : X \to Y \to \mathbb{P}$
- V is polynomial-time computable
- V verifies Q
  - $V\ x\ y \to Q\ x$
  - $Q\ x \to \exists y, V\ x\ y$ and y has polynomial size

[2] Definitions by Fabian

Basic Definitions: Polynomial-Time Reductions[3]

$$P : X \to \mathbb{P} \qquad Q : Y \to \mathbb{P}$$

$$P \preceq_p Q := \exists f : X \to Y, \quad (\forall x, P\ x \leftrightarrow Q(f\ x))$$
$$\wedge \text{ polynomial-time computable } f$$

[3] Definitions by Fabian

Basic Definitions: Polynomial-Time Reductions[3]

$$P : X \to \mathbb{P} \qquad Q : Y \to \mathbb{P}$$

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$$\wedge \text{ polynomial-time computable } f$$
$$\wedge\ f \text{ increases the size polynomially}$$

[3] Definitions by Fabian

Basic Definitions: Polynomial-Time Reductions[3]

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$$\wedge\ f \text{ increases the size polynomially}$$

$$\text{NP-hard } P := \forall Q, Q \in \text{NP} \to Q \preceq_p P$$
$$\text{NP-complete } P := P \in \text{NP} \wedge \text{NP-hard } P$$

[3] Definitions by Fabian

Proving Time Bounds

$$\begin{aligned}
\text{cnf} &:= \mathcal{L}(\text{clause}) \\
\mathcal{E}_N &: \text{assgn} \to \text{cnf} \to \mathbb{B} \\
\mathcal{E}_N\ a\ [\,] &:= \mathsf{T} \\
\mathcal{E}_N\ a\ (C :: N) &:= \mathcal{E}_C\ a\ C\ \&\ \mathcal{E}_N\ a\ N
\end{aligned}$$

1. extraction, running time in terms of arguments

$$\begin{aligned}
T(\mathcal{E}_N)\ a\ [\,] &\geq 9 \\
T(\mathcal{E}_N)\ a\ (C :: N) &\geq T(\mathcal{E}_N)\ a\ N + T(\mathcal{E}_C)\ a\ C + 22
\end{aligned}$$

2. bound by monotonic polynomial $p_{\mathcal{E}_N} : \mathbb{N} \to \mathbb{N}$ s.t.

$$\forall a\ N, T_{\mathcal{E}_N}\ a\ N \leq p_{\mathcal{E}_N}(\|\overline{a}\| + \left\|\overline{N}\right\|)$$

in this case: $p_{\mathcal{E}_N}(n) := n \cdot p_{\mathcal{E}_C}(n) + c_{\mathcal{E}_N} \cdot (n+1)$

#P

counting problem $f : X \to \mathbb{N}$

Verifier Characterisation of #P

$f \in \#\mathrm{P}$, iff there exists a verifier $R : X \to Y \to \mathbb{P}$ s.t.

- R runs in polynomial time
- R only accepts certificates of a polynomial size
- $\forall x, |\{y \mid R\ x\ y\}| = f\ x$

finite cardinalities $|\{y \mid R\ x\ y\}|$ defined using listability

Shortcomings:

- only computable counting problems definable (solution: functional relations)
- precise definition of counting problems requires a bit of care (e.g. **#SAT**)

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