Embedding IEL into IPC

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We consider a fragment of IEL restricted to implication and show that it can be embedded into intuitionistic propositional logic.

This embedding was discovered by **?**, but not formally proven.

Formulas in (the implicational fragment) IPC are generated by the following grammar, we denote the type of formulas with $\mathcal{F}$.

$$\phi, \psi := \psi \to \psi \mid p_i \mid \bot$$

where $i \in \mathbb{N}$. Formulas in the implicational fragment of IEL are generated by the following grammar, the type of formulas is denoted by $\mathcal{F}_{\mathbf{K}}$.

$$\phi, \psi := \psi \to \psi \mid p_i \mid \bot \mid \mathbf{K}\,\phi$$

We use standard natural deduction for both systems. We use the notation $A \vdash \phi$ to denote that $\phi \in \mathcal{F}$ is provable in IPC and the notation $A \vdash_{\mathbf{K}} \phi$ to denote that $\phi \in \mathcal{F}_{\mathbf{K}}$ is provable in IEL.

Embedding

We define an embedding $\mathcal{E} : \mathcal{F}_{\mathbf{K}} \to \mathcal{F}$, mapping IEL to IPC formulas.

$$\begin{aligned} \mathcal{E}\,\mathbf{K}\,\phi &:= (\mathcal{E}\phi \to \mathbf{E}) \to \mathcal{E}\,\phi \\ \mathcal{E}s \to t &:= (\mathcal{E}s) \to (\mathcal{E}t) \\ \mathcal{E}p_i &:= p_i \\ \mathcal{E}\bot &:= \bot \end{aligned}$$

We extent the mapping to contexts by applying $\mathcal{E}$ to every element (as we represent contexts as lists, this corresponds to using the `map` function).

Our main result will be the following theorem: $A \vdash_{\mathbf{K}} \phi \to \mathcal{E}(A) \vdash \mathcal{E}(\phi)$.

Lemma 1. $A \vdash_{\mathbf{K}} \phi \to \mathcal{E}\,(A) \vdash \mathcal{E}\,(\phi)$

Proof. Induction on $A \vdash_{\mathbf{K}} \phi$.

Case ndA: We know $\phi \in A$. By the definition of $\mathcal{E}\,(A)$, $\mathcal{E}\,(\phi) \in \mathcal{E}\,(A)$ follows immediately. We use the assumption rule.

Case ndE: We know $A \vdash_{\mathbf{K}} \bot$ and by the inductive hypothesis $\mathcal{E}\,(A) \vdash \bot$. We can use the ndE rule and are finished.

Case ndIE: We get $\mathcal{E}\,(A) \vdash \mathcal{E}\,(s)$ and $s, \mathcal{E}\,(A) \vdash \mathcal{E}\,(t)$ as inductive hypothesis and can prove $\mathcal{E}\,(A) \vdash \mathcal{E}\,(t)$ by applying the ndIE rule with $\mathcal{E}\,(s)$.

Case ndII: We get $\mathcal{E}\,(s, A) \vdash \mathcal{E}\,(t)$ as an inductive hypothesis and have to show $\mathcal{E}\,(A) \vdash \mathcal{E}\,(s \to t)$. Simplifying $\mathcal{E}$ and applying the ndiII-rule completes the subcase.

Case ndKimp: This case is the hardest one. We need to show $\mathcal{E}(A) \vdash \mathcal{E}(\mathbf{K}\, s \to \mathbf{K}\, t)$ and get $\mathcal{E}(A) \vdash \mathcal{E}(\mathbf{K}\,(s \to t))$ as an inductive hypothesis. By unfolding the definition the IH can be rewritten into $\mathcal{E}(A) \vdash (\mathcal{E}(s \to t) \to \mathbf{E}) \to \mathcal{E}(s \to t)$.

We can unfold the definition of $\mathcal{E}$ in the goal and need to show

$$\mathcal{E}(A) \vdash ((\mathcal{E}(s) \to \mathbf{E}) \to \mathcal{E}(s)) \to ((\mathcal{E}(t) \to \mathbf{E}) \to \mathcal{E}(t))$$

By applying the ndII rule several times, we are left with having to prove $A' \vdash \mathcal{E}(t)$ where $A' := (\mathcal{E}(s) \to \mathbf{E}) \to \mathcal{E}(s), \mathcal{E}(t) \to \mathbf{E}, A$. We first proof $A' \vdash \mathcal{E}(s) \to \mathbf{E}$.

$$\dfrac{\dfrac{\mathcal{E}(s), \mathcal{E}(s \to t), A' \vdash \mathcal{E}(t) \to \mathbf{E} \quad \dfrac{\mathcal{E}(s), \mathcal{E}(s \to t), A' \vdash \mathcal{E}(s) \quad \mathcal{E}(s), \mathcal{E}(s \to t), A' \vdash \mathcal{E}(s) \to \mathcal{E}(t)}{\mathcal{E}(s), \mathcal{E}(s \to t), A' \vdash \mathcal{E}(t)}}{\mathcal{E}(s), \mathcal{E}(s \to t), A' \vdash \mathbf{E}}}{\mathcal{E}(s), A' \vdash (\mathcal{E}(s) \to \mathcal{E}(t)) \to \mathbf{E}}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{E}(s), A' \vdash (\mathcal{E}(s) \to \mathcal{E}(t)) \to \mathbf{E} \quad \dfrac{\text{I.H.}}{A' \vdash ((\mathcal{E}(s) \to \mathcal{E}(t)) \to \mathbf{E}) \to (\mathcal{E}(s) \to \mathcal{E}(t))}}{\mathcal{E}(s), A' \vdash \mathcal{E}(s) \to \mathcal{E}(t)} \quad \dfrac{}{\mathcal{E}(s), A' \vdash \mathcal{E}(s)}}{\mathcal{E}(s), A' \vdash \mathbf{E}}}{A' \vdash \mathcal{E}(s) \to \mathbf{E}}$$

We can now show $A' \vdash \mathcal{E}(s)$.

$$\dfrac{A' \vdash (\mathcal{E}(s) \to \mathbf{E}) \quad A \vdash (\mathcal{E}(s) \to \mathbf{E}) \to \mathcal{E}(s)}{A' \vdash \mathcal{E}(s)}$$

Last we need to prove $A' \vdash \mathcal{E}(s \to t)$.

$$\dfrac{\dfrac{\dfrac{\mathcal{E}(s) \to \mathcal{E}(t), A' \vdash \mathcal{E}(s) \quad \mathcal{E}(s) \to \mathcal{E}(t), A' \vdash \mathcal{E}(s) \to \mathcal{E}(t)}{\mathcal{E}(s) \to \mathcal{E}(t), A' \vdash \mathcal{E}(t)} \quad \mathcal{E}(s) \to \mathcal{E}(t), A' \vdash \mathcal{E}(t) \to \mathbf{E}}{\mathcal{E}(s) \to \mathcal{E}(t), A' \vdash \mathbf{E}}}{A' \vdash (\mathcal{E}(s \to t)) \to \mathbf{E}}$$

We can now finish the proof.

$$\dfrac{\dfrac{A' \vdash ((\mathcal{E}(s) \to \mathcal{E}(t)) \to \mathbf{E}) \to (\mathcal{E}(s) \to \mathcal{E}(t)) \quad A' \vdash (\mathcal{E}(s) \to \mathcal{E}(t)) \to \mathbf{E}}{A' \vdash \mathcal{E}(s) \to \mathcal{E}(t)} \quad A' \vdash \mathcal{E}(s)}{A' \vdash \mathcal{E}(t)}$$

Case ndKbot: We need to show $\mathcal{E}(A) \vdash \neg\neg s$ having $\mathcal{E}(A) \vdash \mathcal{E}(\mathbf{K}\, s)$ as an inductive hypothesis.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s, \neg\mathcal{E}(s), \mathcal{E}(A) \vdash s \quad s, \neg\mathcal{E}(s), \mathcal{E}(A) \vdash \neg\mathcal{E}(s)}{\mathcal{E}(s), \neg\mathcal{E}(s), \mathcal{E}(A) \vdash \bot}}{\mathcal{E}(s), \neg\mathcal{E}(s), \mathcal{E}(A) \vdash \mathbf{E}}}{\neg\mathcal{E}(s), \mathcal{E}(A) \vdash \mathcal{E}(s) \to \mathbf{E}} \quad \dfrac{\text{weakening + I.H.}}{\neg\mathcal{E}(s), \mathcal{E}(A) \vdash (\mathcal{E}(s) \to \mathbf{E}) \to \mathcal{E}(s)}}{\neg\mathcal{E}(s), \mathcal{E}(A) \vdash \mathcal{E}(s)} \quad \dfrac{}{\neg\mathcal{E}(s), \mathcal{E}(A) \vdash \neg\mathcal{E}(s)}}{\neg\mathcal{E}(s), \mathcal{E}(A) \vdash \bot}}{\mathcal{E}(A) \vdash \neg\neg\mathcal{E}(s)}$$

Case ndKpos: We get $\mathcal{E}(A) \vdash \mathcal{E}(s)$ as an inductive hypothesis and need to prove $\mathcal{E}(A) \vdash \mathcal{E}(\mathbf{K}\, s)$. Thus by the definition of $\mathcal{E}$ we need to show $\mathcal{E}(A) \vdash (\mathcal{E}(s) \to \mathbf{E}) \to \mathcal{E}(s)$.

$$\dfrac{\dfrac{\mathcal{E}(a) \vdash \mathcal{E}(s)}{\mathcal{E}(s) \to \mathbf{E}), \mathcal{E}(A) \vdash \mathcal{E}(s)}}{\mathcal{E}(A) \vdash (\mathcal{E}(s) \to \mathbf{E}) \vdash \mathcal{E}(s)}$$

The cases for disjunction and conjucntion are easy and solvable by unfolding the definition of $\mathcal{E}$ and applying the inductive hypothesis. □

We thus obtain a consistency proof for IEL (and IEL^- using the consistency of IPC.

Lemma 2. *IEL is consistent.*

Proof. If IEL was not consistent i.e. $\vdash_{\mathbf{K}} \bot$, then $\vdash \mathcal{E}(\bot)$ and thus $\vdash \bot$. This would contradict the consistency of IPC. □

An interesting open question is if the converse direction is also true (and provable).