# Peano Arithmetic in Constructive Type Theory and Tennenbaum's Theorem


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- This work is build on the Coq library of undecidability proofs [Forster et al., 2020]
- Followed classical presentations of Tennenbaum's Theorem [Smith, 2014]
- I want to thank Prof. Dr. Weber, Dominik Kirst, Prof. Dr. Smolka.
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# Questions of the Talk

What are the computable models of Peano arithmetic?

Why care about constructive mathematics?

1. Setup → 2. Constr. Math -- Break --> 3. Tennenbaum
(Tennenbaum) again

# 1. Setup

# Peano Arithmetic

We have a first-order language.

- Logical symbols:

$$\land, \lor, \to, \neg, =, \forall, \exists$$

- Non-logical symbols:

$$O, S, \oplus, \otimes$$

And Axioms:

$$\begin{aligned} Sx &\neq O & Sx = Sy &\to x = y \\ x \oplus O &= x & (Sx) \oplus y &= S(x \oplus y) \\ x \otimes O &= O & (Sx) \times y &= y \oplus (x \otimes y) \end{aligned}$$

$$\varphi(O) \to (\forall x.\, \varphi(x) \to \varphi(Sx)) \to \forall x.\, \varphi(x)$$

The intended interpretation is over $\mathbb{N}$

interprete

| | $\mathbb{N}$ |
|---|---|
| $0$ | $0$ |
| $S$ | $n \mapsto n+1 \quad : \mathbb{N} \to \mathbb{N}$ |
| $\oplus$ | $(x,y) \mapsto x+y \quad : \mathbb{N}^2 \to \mathbb{N}$ |
| $\otimes$ | $(x,y) \mapsto x \cdot y \quad : \mathbb{N}^2 \to \mathbb{N}$ |

# Models of PA

$0$

$S$

$\oplus$

$\otimes$

interpre

# Non-standard Model of PA

## Compactness Theorem

If every finite $\Gamma \subseteq \mathcal{T}$ has a model, then so has $\mathcal{T}$.

# Models of PA

# Computable Models

- There is a bijection $g : \mathbb{N} \to \mathcal{M}$
- The operations on $\mathcal{M}$ are computable.

**Tennebaum's Theorem (classical) [Tennenbaum, 1959, Kaye, 2011, Smith, 2014]**

No countable non-standard model of PA is computable.

Still: what do we mean by computable?

# Computable Models

One way to make the statement precise is to replace computable with “computable by a Turing machine”.

Definitions and results involving computability:

- $A \subseteq \mathbb{N}$ is decidable $:\Leftrightarrow$ there is a computable $f$ halting on every input with $f(n) = 1 \leftrightarrow n \in A$.
- $A \subseteq \mathbb{N}$ is enumerable $:\Leftrightarrow$ there is a computable $f$ such that $f(n)$ halts iff $n \in A$.
- Theorem: if $f, g$ are computable, then $f \circ g$ is computable.

# Halting Problem

Given the set of all Turing Machines TM we can define $\mathcal{O}\colon \mathrm{TM} \to \{0, 1, *\}$ by

$$\mathcal{O}(T) := \begin{cases} 1, & T \text{ halts on empty input} \\ 0, & T \text{ does not halt on empty input} \\ *, & \text{else} \end{cases}$$

Is $\mathcal{O}$ defined for all Turing machines? For every given $T$, we have

- $T$ halts, therefore $\mathcal{O}$ returns 1
- $T$ does not halt, therefore $\mathcal{O}$ returning 0

This shows that $\mathcal{O} : \mathrm{TM} \to \{0, 1\}$ is a function, and since it solves the Halting Problem, it is not computable.

# 2. Constructive Mathematics

# Constructive Mathematics

Mathematics without the law of excluded middle (LEM).

Removing axioms is very common in mathematics:

# Freedom

Without LEM we have an agnostic meta-theory regarding computability

# Constructive Mathematics

- Constructive mathematics started with [Brouwer, 1907].
- Contributors: Heyting, Kolmogorov, Markov, Bishop, Weyl, Friedman ...
- Appears naturally as the internal logic of some topoi.
- Important applications in computer science.
- Recent AMS price awarded to a introductory paper by [Bauer, 2017].

Things to get used to:

- No more truth-semantics.
- In general, double-negations stick around.
- Showing $\neg\forall x.\, P(x)$ does not prove $\exists x.\, \neg P(x)$.
- There are several notions of finite and infinite.
- Possible to assume that all functions $\mathbb{R} \to \mathbb{R}$ are continuous.

# Intermission

Break for discussion

# 3. Tennenbaum's Theorem

# Overview again

# Notions in Computatbility

For any predicate $P$ on $\mathbb{N}$, it is

- decidable iff $\exists(f{:}\mathbb{N} \to \mathbb{N})\, \forall n.\, (P(n) \leftrightarrow f(n) = 0)$.
- enumerable iff $\exists(f : \mathbb{N} \to \mathbb{N} \cup \{*\})\, \forall n.\, (P(n) \leftrightarrow \exists m.\, f(m) = n)$.

Any type $X$ is

- discrete iff equality $=$ on $X$ is decidable.
- apart iff apartness $\neq$ in $X$ is decidable.
- enumerable iff $\exists(f{:}\mathbb{N} \to X \cup \{*\})\, \forall x\, \exists n.\, f(n) = x$.
- witnessing iff X has a witness operator.

# Q-Representability

Classical result (c.f. [Smith, 2013]):

### Q-Representability

If $f : \mathbb{N} \to \mathbb{N}$ is (Turing) computable, then there is a binary $\Sigma_1$ formula $\varphi_f(x, y)$ such that for every $n : \mathbb{N}$

$$Q \vdash \forall y.\, \varphi_f(\overline{n}, y) \leftrightarrow \overline{f(n)} = y$$

By CT, every function $\mathbb{N} \to \mathbb{N}$ is (Turing) computable, so we get:

### $\mathsf{CT}_Q$

Every function $f : \mathbb{N} \to \mathbb{N}$ is representable in Q by some binary $\Sigma_1$ formula $\varphi_f$.

# Representibility Theorem

**Representibility Theorem ($RT_Q$)**

Any enumerable (decidable) predicate on $\mathbb{N}$ is weakly (strongly) representable in Q.

# Overview again

# Result in Computability Theory

## Inseparable enumerable predicates

Assuming $\mathsf{RT}_Q$, there are disjoint enumerable predicates $A, B$ on $\mathbb{N}$ which cannot be separated by a decidable predicate $C$.

# Tennenbaum ingredients

## Inseparable Formulas (Insep)

Assuming $\mathrm{RT}_{\mathrm{Q}}$, there are disjoint $\Sigma_1$-formulas which cannot be separated by a decidable predicate $D$ on $\mathbb{N}$.

## Markov's Principle (MP)

If a function $\mathbb{N} \to \mathbb{N}$ potentially hits 0, it hits 0.

## Lemma

Assuming $\mathrm{RT}_{\mathrm{Q}}$ and MP, if $\mathcal{M} \not\approx \mathbb{N}$ then there potentially is an element $d : \mathcal{M}$ such that $\{ n : \mathbb{N} \mid n \text{ divides } d \}$ is not decidable.

## Tennenbaum's Theorem (constructive)

Assuming $\mathsf{RT}_\mathsf{Q}$ and MP, if $\mathcal{M}$ has decidable divisibility with respect to numerals, then $\mathcal{M} \simeq \mathbb{N}$.

# Tennenbaum

# Tennenbaum

Wrapping things up

# Coq Development

Coq is a proof-assistant. Helps by organizing and keeping track of proofs, some automation and most crucially: checks them for correctness.

- Mechanized first-order PA, building on work from the Coq library of undecidability proofs [Forster et al., 2020]
- 4200 lines of code (loc)
- 1100 loc for specifications
- 3200 loc of proofs.
- The equality $=$ of PA was interpreted as equality on the domain type.

## Contributions

- Showing a finite fragment of PA undecidable [Kirst and Hermes, 2021].
- To the best of knowledge: first mechanized proof of Tennenbaum's Theorem.
- Discussion and mechanization of two more variants of the theorem. [Makholm, , McCarty, 1987, McCarty, 1988]

## Future Work

- Deductive proves of some more work intensive theorems
- Express computability of operations with respect to an abstract notion of computability.
- Equivalence to some version of MP?
- Mechanize the proof of $\mathsf{CT}_Q \to \mathsf{RT}_\mathsf{Q}$ (seems very reachable)
- Show $\mathsf{CT} \to \mathsf{CT}_\mathsf{Q}$ (ambitious)

# Final Words

*"Dieses Tertium non datur dem Mathematiker zu nehmen, wäre etwa, wie wenn man dem Astronomen das Fernrohr oder dem Boxer den Gebrauch der Fäuste untersagen wollte."*
(Hilbert)

*"Restricting a mathematician to a logic with the law of excluded middle is like putting boxing gloves over the hands of a boxer. It greatly restricts what he can do."*

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# Variants of Tennenbaum

Both variants need a stronger version of inseparable formulas $\alpha, \beta$. They are called HA-inseparable iff $\Sigma_1$ with $\mathsf{HA} \vdash_i \neg\exists x.\, \alpha(x) \wedge \beta(x)$. Given the existence of such formulas, we have:

## Makholm (constructive)

Every enumerable and discrete $\mathcal{M} \vDash \mathsf{PA}$ has $\forall e{:}\,\mathcal{M}.\, \neg\neg \mathsf{std}\, e$. [Makholm, ]

## McCarty

Under a more constructive Tarski semantics, HA is categorical. [McCarty, 1987, McCarty, 1988]