Undecidability of Peano Arithmetic

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SAARLAND UNIVERSITY

COMPUTER SCIENCE

SIC Saarland Informatics Campus

Main Statement (Informally)

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Considering logical entailment for the first-order theory of Peano Arithmetic (PA), we can show

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The proof of this is fully mechanised Coq.

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- Hilbert's 10th Problem [Larchey-Wendling and Forster, 2019]

Coq and Mathematics

Mathematics is in the most part implicitly framed in set theory. Coq is based on a different kind of foundational theory. (dependent type theory)

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- $2 = (0, 0)$
- $\emptyset + 1 = \{\emptyset\}$
- $\sin(\cos) \in \pi$

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There are a lot of intuitions mathematicians have, which are not justified in set theory, but *are* when using a type theory.

- $2 = \{\emptyset, \{\emptyset\}\} = (\emptyset, \emptyset) = (0, 0)$
- $0 + 1 = 1$
- most likely $\sin(\cos) \notin \pi$

Coq and Mathematics

Mathematics is in the most part implicitly framed in set theory. Coq is based on a different kind of foundational theory. (dependent type theory)

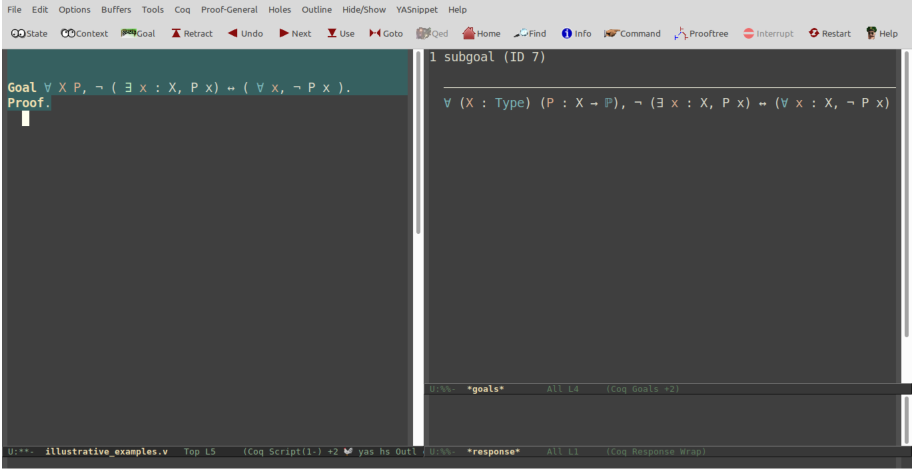
There are a lot of intuitions mathematicians have, which are not justified in set theory, but *are* when using a type theory.

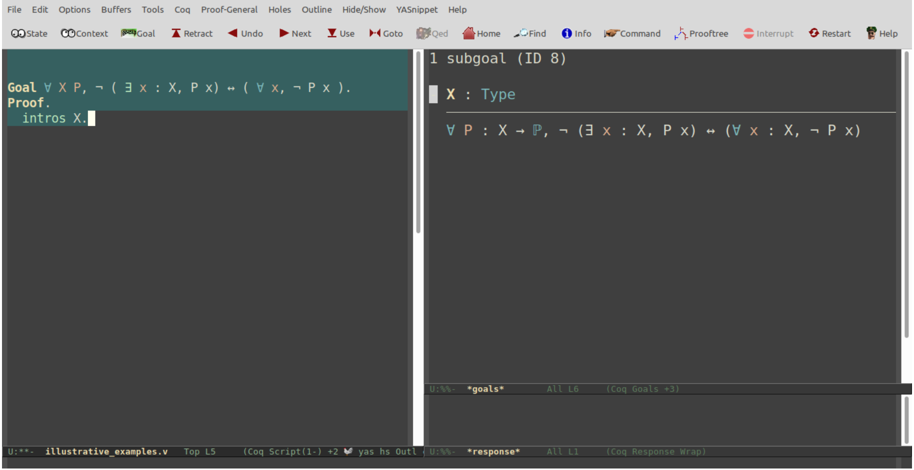
- $2 = (0, 0)$
- $\emptyset + 1 = \{\emptyset\}$
- $\text{sin}(\text{cos}) \in \pi$

In agreement with intuition, the above statements do not make sense in type theory!

Proofs in Coq

Let's look at some proofs inside of Coq!


File Edit Options Buffers Tools Coq Proof-General Holes Outline Hide/Show YASnippet Help
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
1 subgoal (ID 7)
∀ (X : Type) (P : X → ℙ), ¬ (∃ x : X, P x) ↔ (∀ x : X, ¬ P x)
U:%%- *goals* All L4 (Coq Goals +2)
U:**- illustrative_examples.v Top L5 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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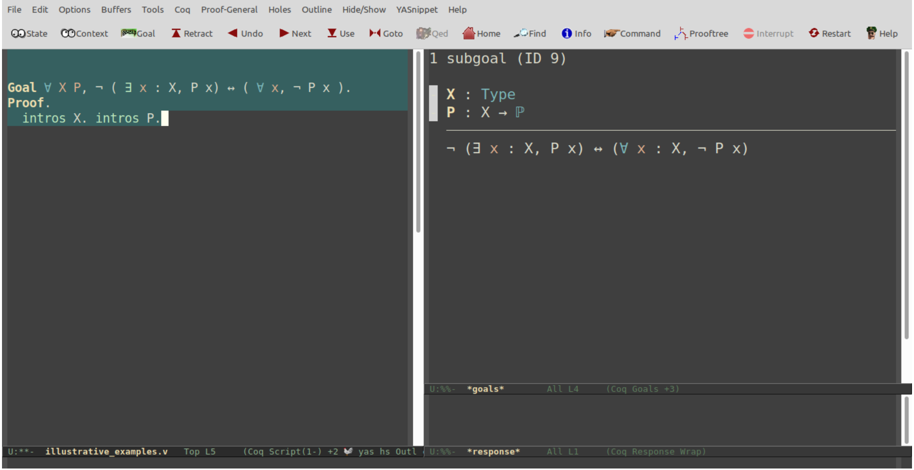
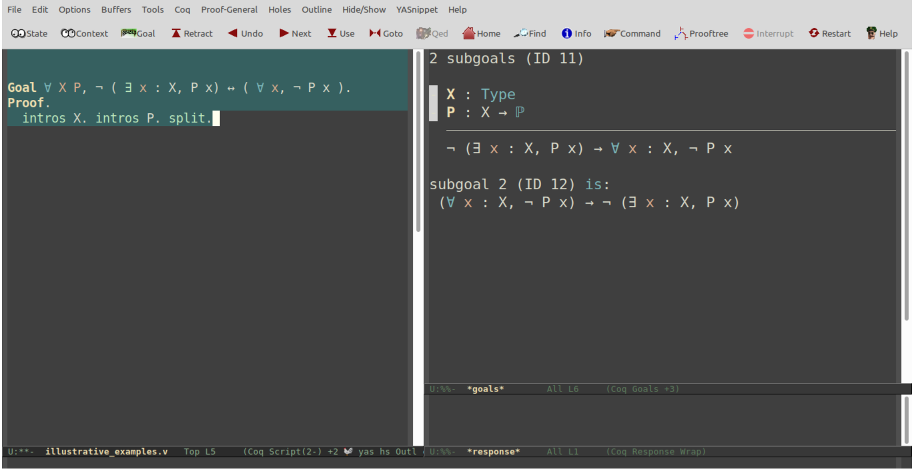
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X.
1 subgoal (ID 8)
X : Type
∀ P : X → ℙ, ¬ (∃ x : X, P x) ↔ (∀ x : X, ¬ P x)
U:%%- *goals* All L6 (Coq Goals +3)
U:**- illustrative_examples.v Top L5 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)


File Edit Options Buffers Tools Coq Proof-General Holes Outline Hide/Show YASnippet Help
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P.
1 subgoal (ID 9)
X : Type
P : X → ℙ
¬ (∃ x : X, P x) ↔ (∀ x : X, ¬ P x)
U:%%- *goals* All L4 (Coq Goals +3)
U:**- illustrative_examples.v Top L5 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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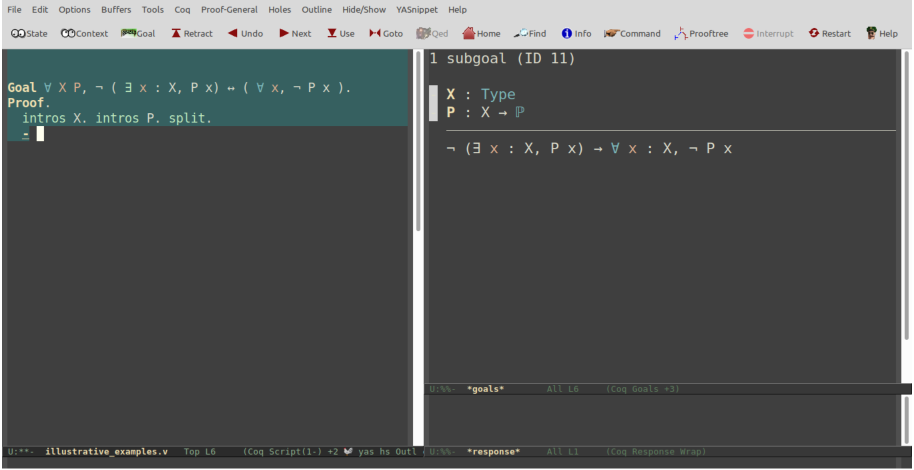
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
2 subgoals (ID 11)
X : Type
P : X → ℙ
¬ (∃ x : X, P x) → ∀ x : X, ¬ P x
subgoal 2 (ID 12) is:
(∀ x : X, ¬ P x) → ¬ (∃ x : X, P x)
U:%%- *goals* All L6 (Coq Goals +3)
U:**- illustrative_examples.v Top L5 (Coq Script(2-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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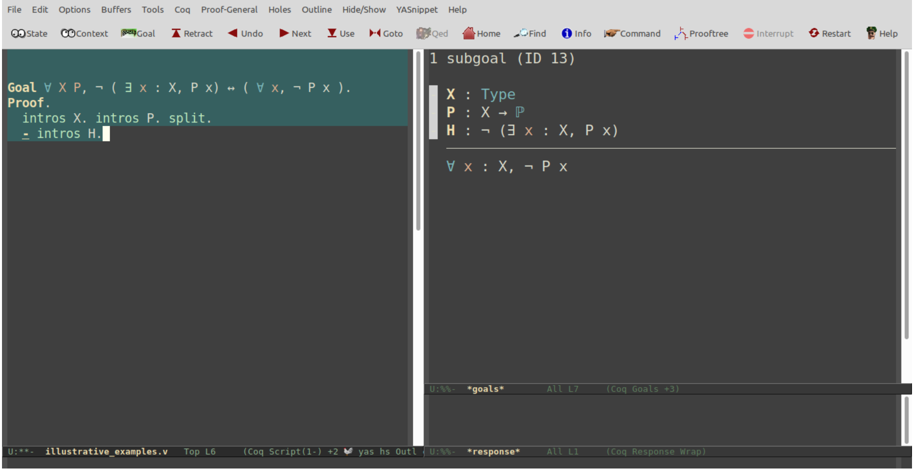
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
-
1 subgoal (ID 11)
X : Type
P : X → ℙ
¬ (∃ x : X, P x) → ∀ x : X, ¬ P x
U:%%- *goals* All L6 (Coq Goals +3)
U:**- illustrative_examples.v Top L6 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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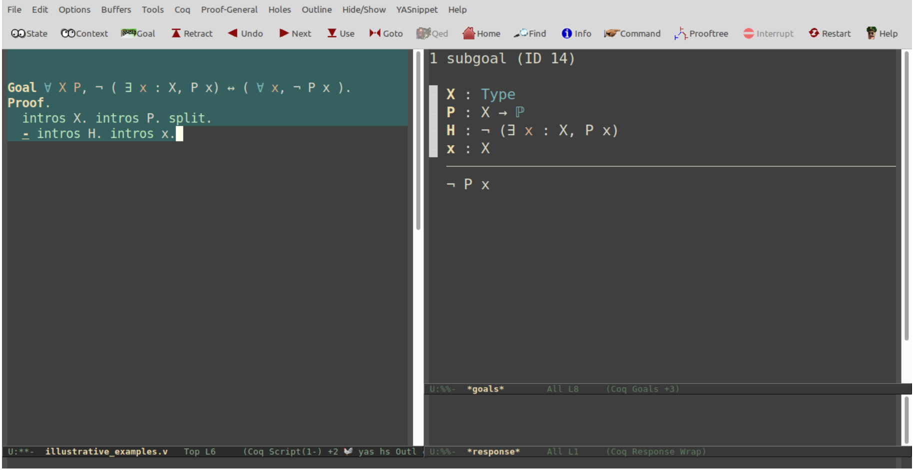
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H.
1 subgoal (ID 13)
X : Type
P : X → ℙ
H : ¬ (∃ x : X, P x)
∀ x : X, ¬ P x
U:%%- *goals* All L7 (Coq Goals +3)
U:**- illustrative_examples.v Top L6 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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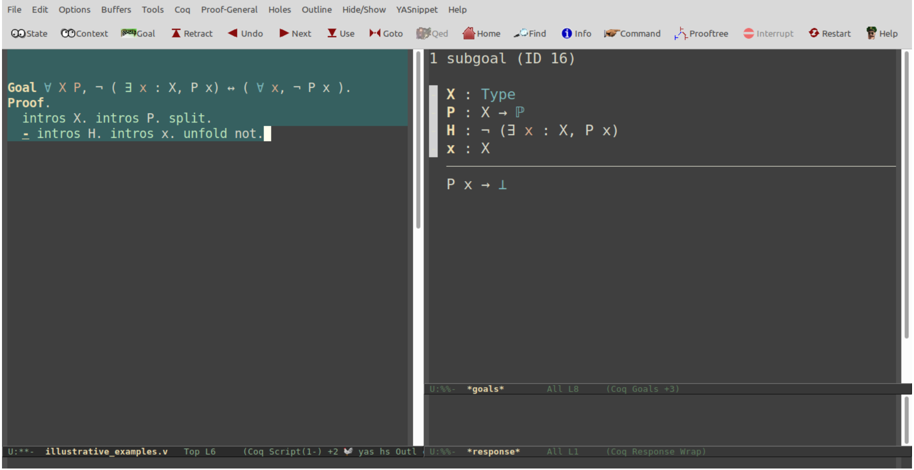
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x.
1 subgoal (ID 14)
X : Type
P : X → ℙ
H : ¬ (∃ x : X, P x)
x : X
¬ P x
U:%%- *goals* All L8 (Coq Goals +3)
U:**- illustrative_examples.v Top L6 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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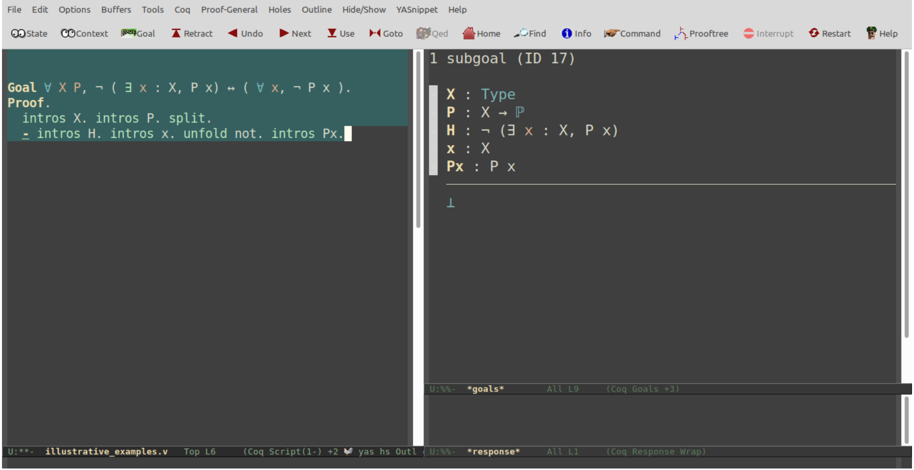
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not.
1 subgoal (ID 16)
X : Type
P : X → ℙ
H : ¬ (∃ x : X, P x)
x : X
P x → ⊥
U:%%- *goals* All L8 (Coq Goals +3)
U:**- illustrative_examples.v Top L6 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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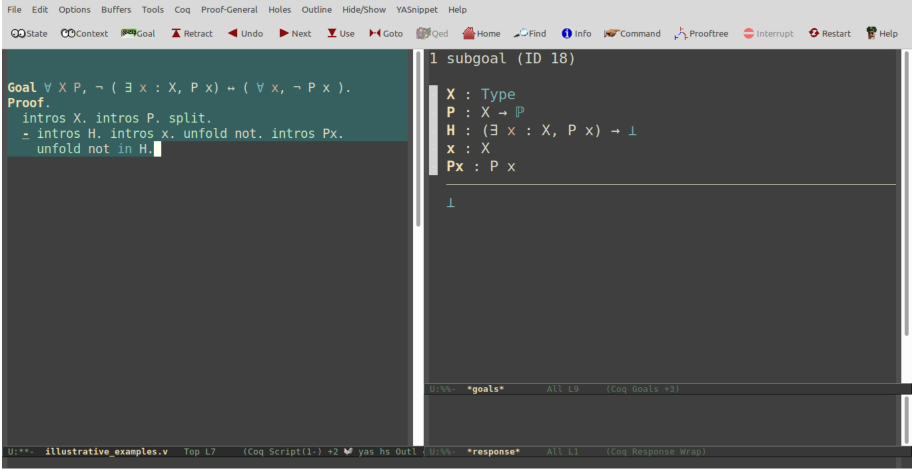
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
1 subgoal (ID 17)
X : Type
P : X → ℙ
H : ¬ (∃ x : X, P x)
x : X
Px : P x
⊥
U:%%- *goals* All L9 (Coq Goals +3)
U:**- illustrative_examples.v Top L6 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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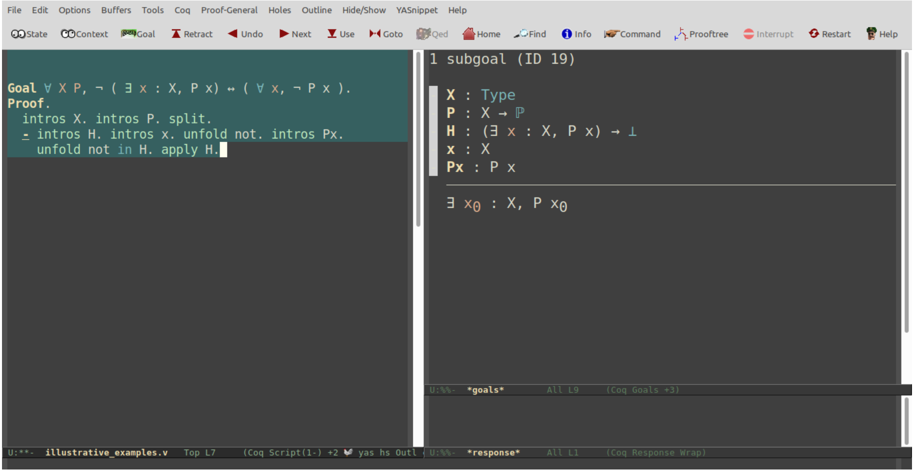
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H.
1 subgoal (ID 18)
X : Type
P : X → ℙ
H : (∃ x : X, P x) → ⊥
x : X
Px : P x
⊥
U:%%- *goals* All L9 (Coq Goals +3)
U:**- illustrative_examples.v Top L7 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)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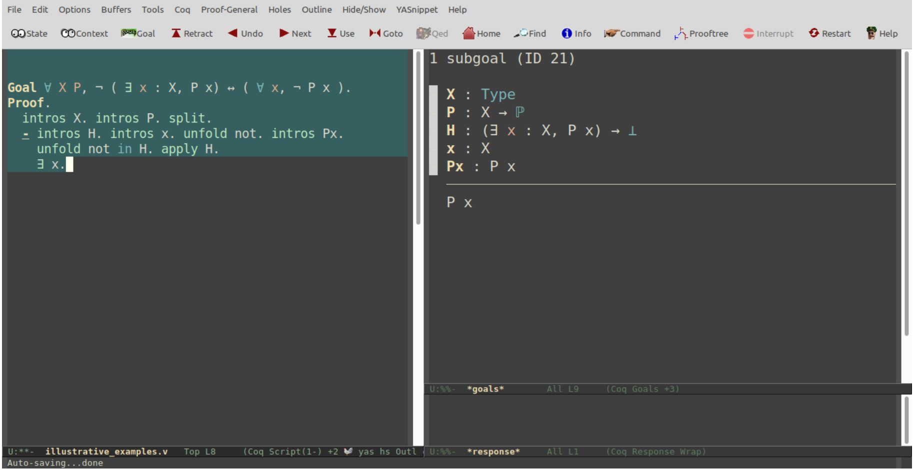
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
U:**- illustrative_examples.v Top L7 (Coq Script(1-) +2 yas hs Outl
1 subgoal (ID 19)
X : Type
P : X → ℙ
H : (∃ x : X, P x) → ⊥
x : X
Px : P x
∃ x0 : X, P x0
U:%%- *goals* All L9 (Coq Goals +3)
U:%%- *response* All L1 (Coq Response Wrap)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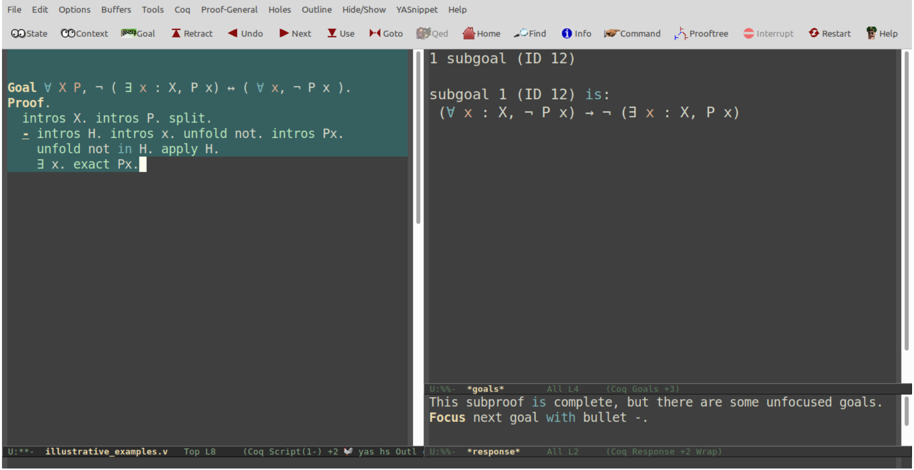
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x.
1 subgoal (ID 21)
X : Type
P : X → ℙ
H : (∃ x : X, P x) → ⊥
x : X
Px : P x
P x
U:%%- *goals* All L9 (Coq Goals +3)
U:**- illustrative_examples.v Top L8 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)
Auto-saving...done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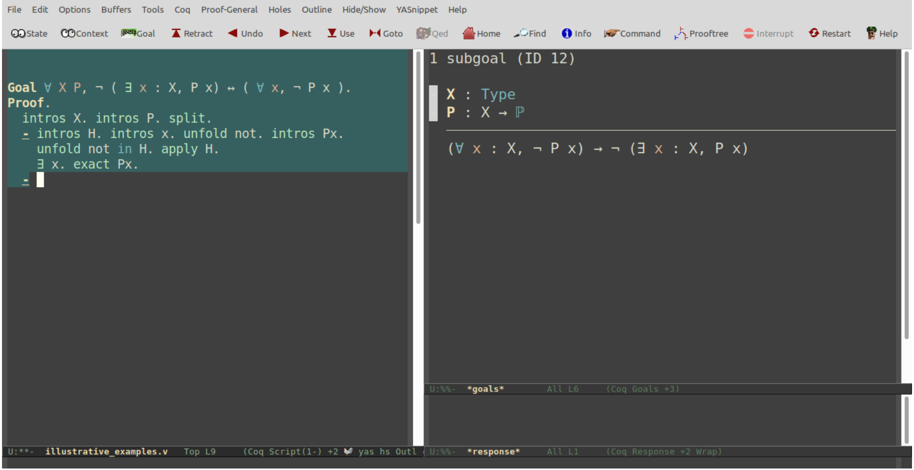
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
U:**- illustrative_examples.v Top L8 (Coq Script(1-) +2 yas hs Outl
1 subgoal (ID 12)
subgoal 1 (ID 12) is:
(∀ x : X, ¬ P x) → ¬ (∃ x : X, P x)
U:%%- *goals* All L4 (Coq Goals +3)
This subproof is complete, but there are some unfocused goals.
Focus next goal with bullet -.
U:%%- *response* All L2 (Coq Response +2 Wrap)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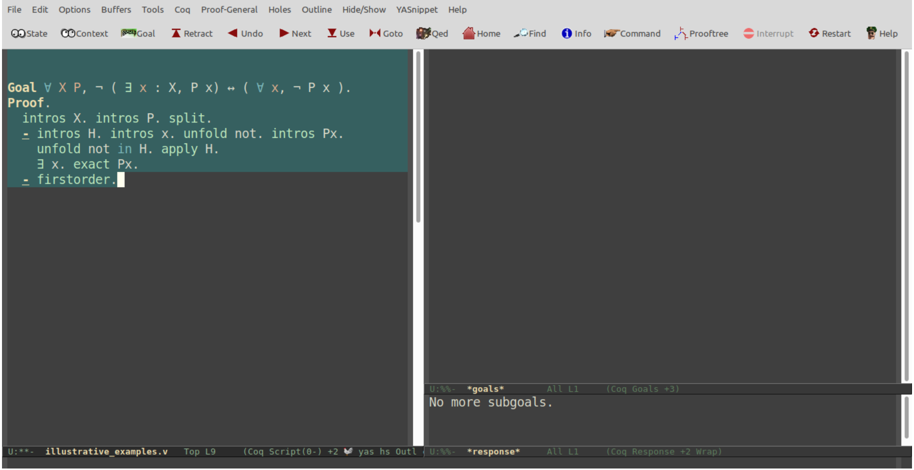
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
-
1 subgoal (ID 12)
X : Type
P : X → ℙ
(∀ x : X, ¬ P x) → ¬ (∃ x : X, P x)
U:%%- *goals* All L6 (Coq Goals +3)
U:**- illustrative_examples.v Top L9 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response +2 Wrap)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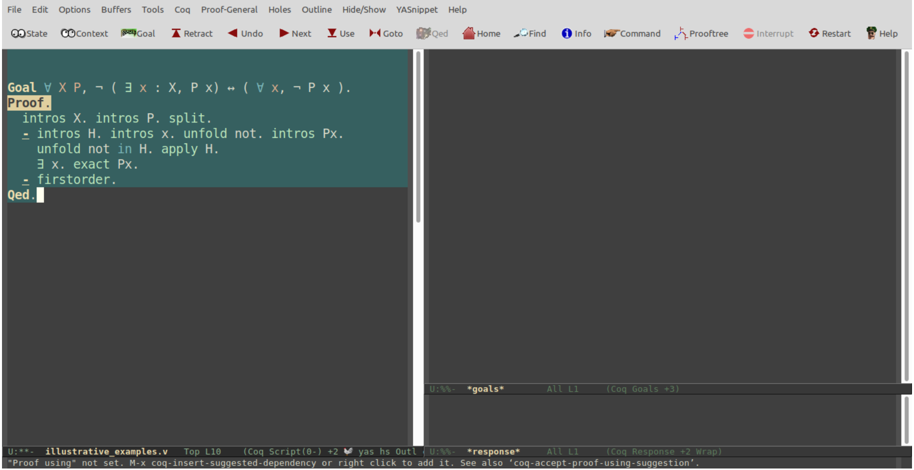
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
- firstorder.
No more subgoals.
U:**- illustrative_examples.v Top L9 (Coq Script(0-) +2 yas hs Outl
U:%%- *goals* All L1 (Coq Goals +3)
U:%%- *response* All L1 (Coq Response +2 Wrap)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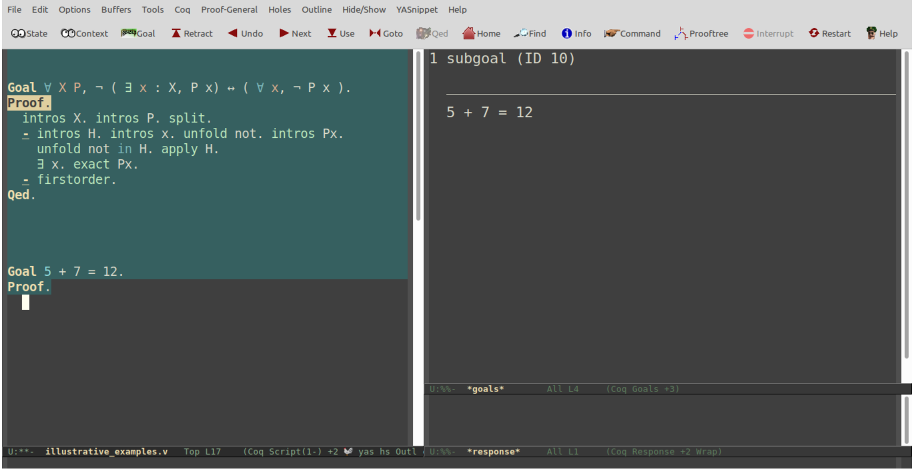
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
- firstorder.
Qed.
U:%%- *goals* All L1 (Coq Goals +3)
U:**- illustrative_examples.v Top L10 (Coq Script(0-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response +2 Wrap)
"Proof using" not set. M-x coq-insert-suggested-dependency or right click to add it. See also ‘coq-accept-proof-using-suggestion’.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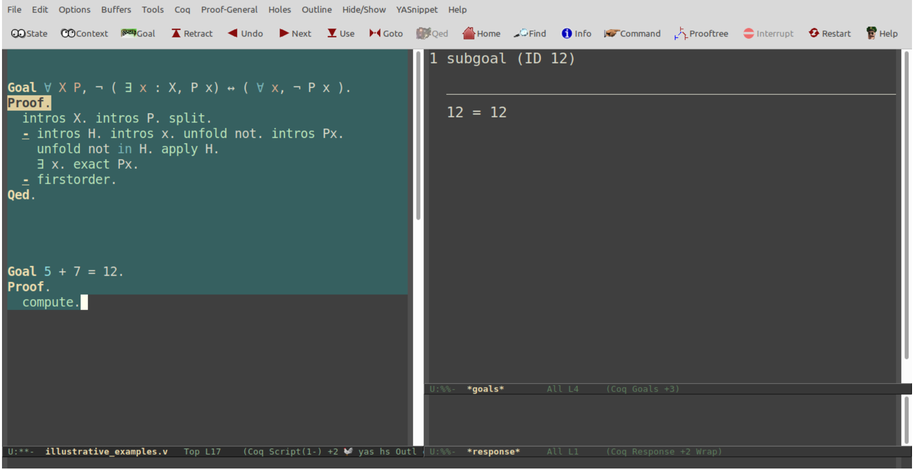
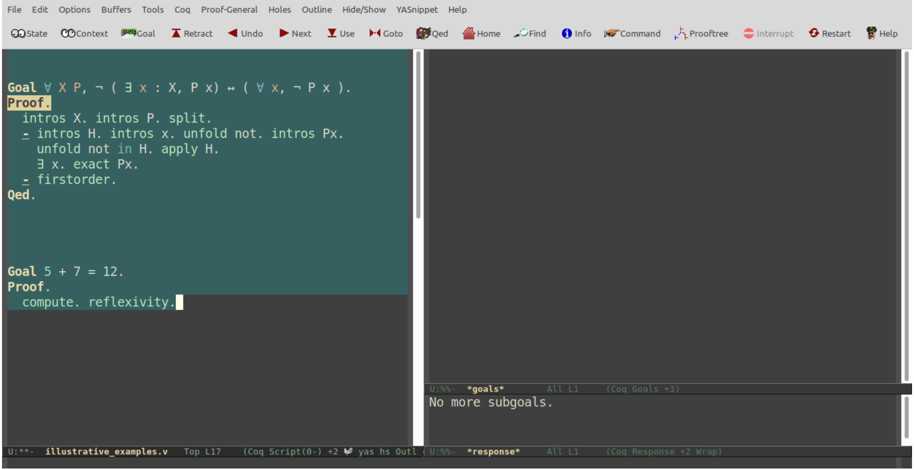
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
- firstorder.
Qed.
Goal 5 + 7 = 12.
Proof.
1 subgoal (ID 10)
5 + 7 = 12
U:%%- *goals* All L4 (Coq Goals +3)
U:**- illustrative_examples.v Top L17 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response +2 Wrap)


File Edit Options Buffers Tools Coq Proof-General Holes Outline Hide/Show YASnippet Help
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
- firstorder.
Qed.
Goal 5 + 7 = 12.
Proof.
compute.
1 subgoal (ID 12)
12 = 12
U:%%- *goals* All L4 (Coq Goals +3)
U:**- illustrative_examples.v Top L17 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response +2 Wrap)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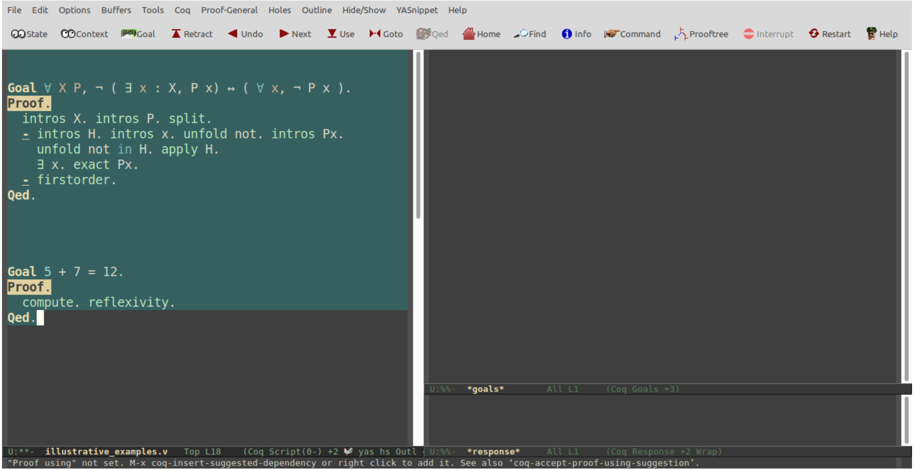
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
- firstorder.
Qed.
Goal 5 + 7 = 12.
Proof.
compute. reflexivity.
U:**- illustrative_examples.v Top L17 (Coq Script(0-) +2 yas hs Outl
U:%%- *goals* All L1 (Coq Goals +3)
No more subgoals.
U:%%- *response* All L1 (Coq Response +2 Wrap)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Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X. intros P. split.
- intros H. intros x. unfold not. intros Px.
unfold not in H. apply H.
∃ x. exact Px.
- firstorder.
Qed.
Goal 5 + 7 = 12.
Proof.
compute. reflexivity.
Qed.
U:%%- *goals* All L1 (Coq Goals +3)
U:**- illustrative_examples.v Top L18 (Coq Script(0-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response +2 Wrap)
"Proof using" not set. M-x coq-insert-suggested-dependency or right click to add it. See also 'coq-accept-proof-using-suggestion'.

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Intuition

β decidable by algorithm and f computable $\to$ $(\beta \circ f \leftrightarrow \alpha)$ decidable. $\lightning$

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- Every function definable in Coq is computable

Reductions

Relevant for us: What are A, B, α, β and f in our case?

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Fragment FA of Peano Arithmetic

The first-order theory of PA has the following symbols:

Function Symbols : $0 \quad S \quad + \quad \cdot$ **Predicate Symbols :** $\equiv$

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\begin{aligned}
\textbf{Zero addition} &: \forall x.\ 0 + x \equiv x \\
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We define expressions containing variables, we call

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We call a list $L = [e_1, \ldots, e_n]$ of atomic equations e_j a H10 problem and extend $[\ \ ]_\sigma$ to problems by $[L]_\sigma := [e_1]_\sigma \land \ldots \land [e_n]_\sigma$.

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Given a H10 problem L, we can now ask the question:

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Can L be satisfied? $\leftrightarrow$ Can we show $\exists\sigma.\ [L]_\sigma$?

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$$L = [x + x = y \ , \ y \cdot y = x]$$

We want to send this to a formula in FA which intuitively expresses the satisfiability of L.

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Examples

- $\mathbb{N} \vDash (x_1 + x_2 \equiv x_3) \;=\; \forall \rho.\;\; \rho(1) + \rho(2) = \rho(3)$
- $\mathbb{N} \vDash (\forall x.\, 0 + x \equiv x) \;=\; \forall n : \mathbb{N}.\, 0 + n = n$

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If we replace $\mathbb{N}$ with some other domain D providing

- $\mathbb{O} : D$
- $\mathbb{S} : D \to D$
- $\oplus : D \times D \to D$
- $\otimes : D \times D \to D$

we get the more general notion of a model $(D, \mathbb{O}, \mathbb{S}, \oplus, \otimes)$ for arithmetic.

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For the proof of these equations we need the axioms we assumed for FA.

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Thank you for your attention!

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