Undecidability of Peano Arithmetic

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Main Statement (Informally)

In the context of the first-order theory of Peano Arithmetic (PA), we can ask the

Question: Is there an algorithm which can tell us for every formula φ if it holds in every PA model?

Answer: No

The proof of this has been fully mechanised in Coq.

The Coq Proof Assistant

- Coq is an interactive proof assistant [The Coq Proof Assistant, 2020]
- Based on the *calculus of constructions* by Thierry Coquand
- Work started in 1984 by Coquand and Gérard Huet
- Is still actively developed and supported

Noteworthy proofs that have been mechanised in Coq:

- Four Colour Theorem [Gonthier, 2008]
- Feit-Thompson Theorem [Gonthier et al., 2013]
- CompCert Compiler [Leroy et al., 2012]

and most relevant for this talk:

- Hilbert's 10th Problem [Larchey-Wendling and Forster, 2019]

Coq and Mathematics

Mathematics is in the most part implicitly framed in set theory. Coq is based on a different kind of foundational theory. (dependent type theory)

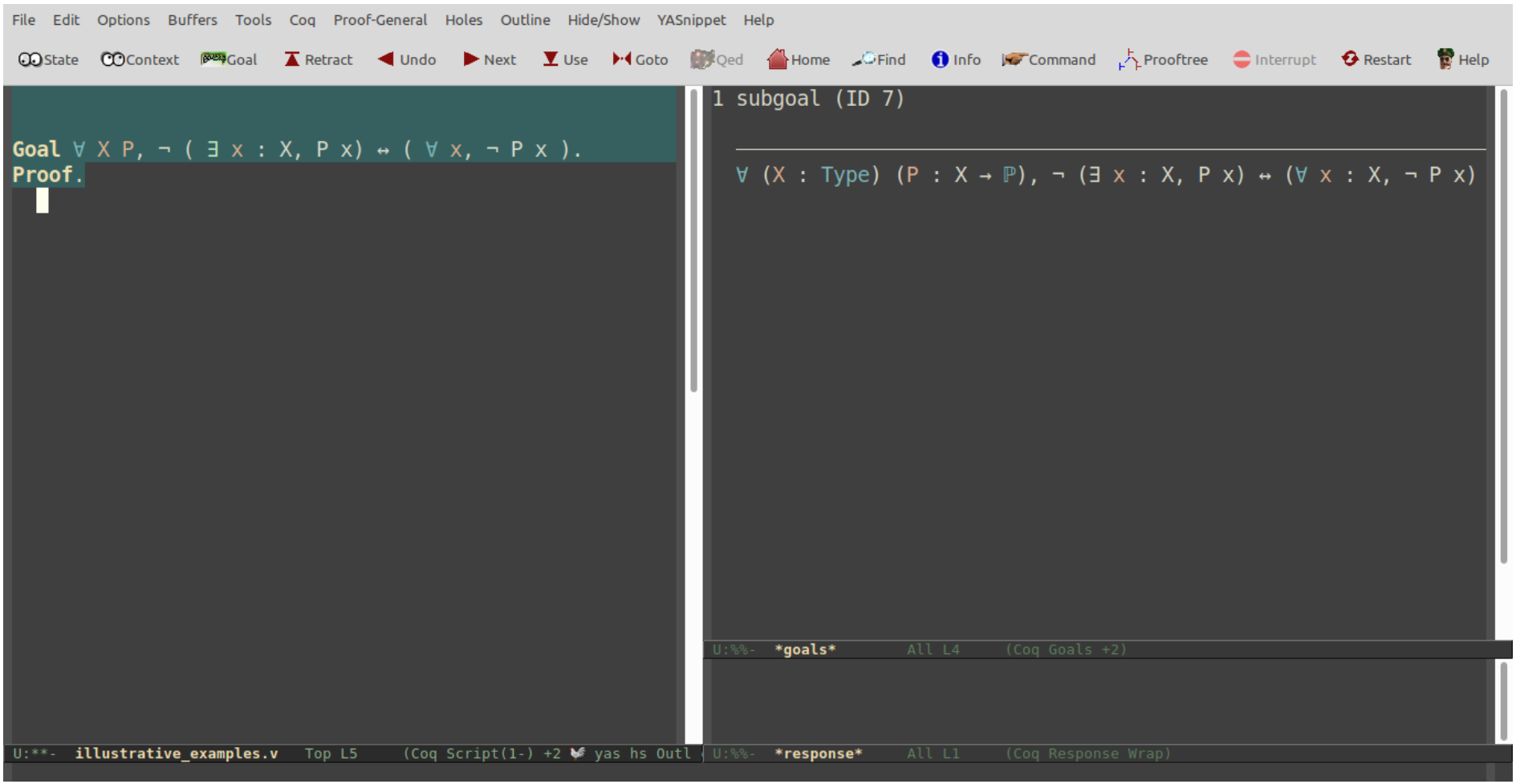
There are a lot of intuitions mathematicians have, which are not justified in set theory, but *are* when using a type theory.

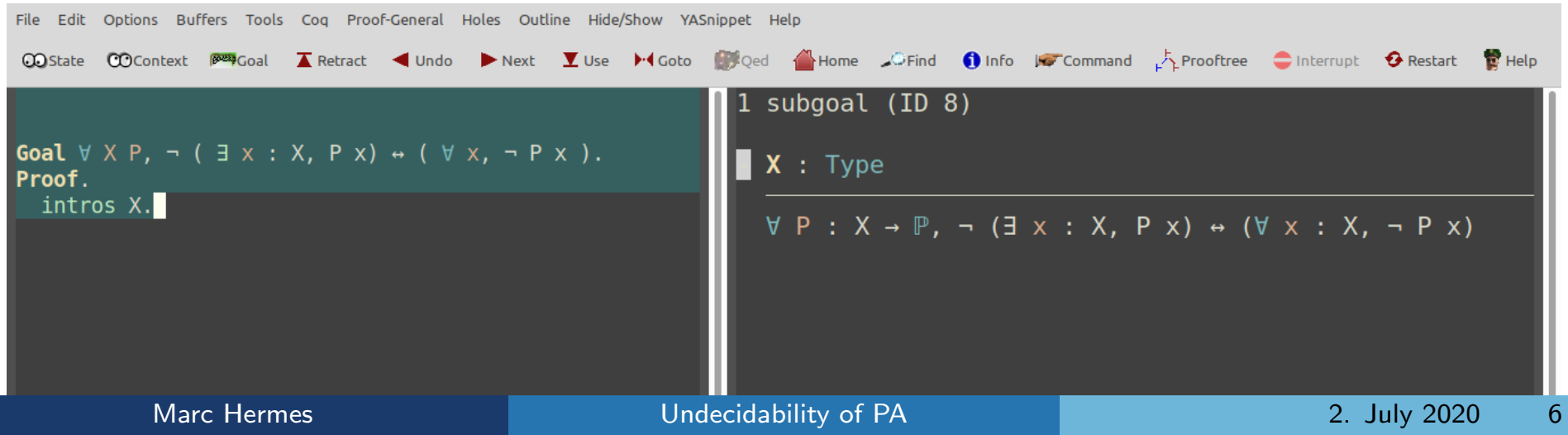
- $2 = (0, 0)$
- $\emptyset + 1 = \{\emptyset\}$
- $\sin(\cos) \in \pi$

In agreement with intuition, the above statements do not make sense in type theory!

Proofs in Coq

Let's look at some proofs inside of Coq!


File Edit Options Buffers Tools Coq Proof-General Holes Outline Hide/Show YASnippet Help
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
1 subgoal (ID 7)
∀ (X : Type) (P : X → ℙ), ¬ (∃ x : X, P x) ↔ (∀ x : X, ¬ P x)
U:%%- *goals* All L4 (Coq Goals +2)
U:**- illustrative_examples.v Top L5 (Coq Script(1-) +2 yas hs Outl
U:%%- *response* All L1 (Coq Response Wrap)


File Edit Options Buffers Tools Coq Proof-General Holes Outline Hide/Show YASnippet Help
State Context Goal Retract Undo Next Use Goto Qed Home Find Info Command Prooftree Interrupt Restart Help
Goal ∀ X P, ¬ (∃ x : X, P x) ↔ (∀ x, ¬ P x).
Proof.
intros X.
1 subgoal (ID 8)
X : Type
∀ P : X → ℙ, ¬ (∃ x : X, P x) ↔ (∀ x : X, ¬ P x)

Undecidability along Reductions

Undecidable Predicate (informally)

A predicate which has no algorithmic decision procedure.

Let α be some **undecidable** predicate on a type A and β a predicate on B. If we have a **computable** function $f : A \to B$ with

$$\forall x : A.\ \ \alpha(x) \ \leftrightarrow\ \ \beta(f(x))$$

then β is also undecidable.

Intuition

β decidable by algorithm and f computable $\to$ $(\beta \circ f \leftrightarrow \alpha)$ decidable. ↯

Reductions

Definition

Let α be some predicate on a type A and β a predicate on B. Then we call $f : A \to B$ a reduction from α to β iff

$$\forall x : A.\ \ \alpha(x) \ \leftrightarrow\ \ \beta(f(x))$$

and f is **computable**. ~~and f is computable~~.

The above gives a synthetic notion for reductions, which is justified by noting that from the outside we can recognise:

- Coq's internal logic is constructive
- Every function definable in Coq is computable

Reductions

Relevant for us: What are A, B, α, β and f in our case?

$$f : A \to B \quad \text{s.t.} \quad \forall x.\ \ \alpha(x) \ \leftrightarrow \ \beta(f(x))$$

Fragment FA of Peano Arithmetic

The first-order theory of PA has the following symbols:

Function Symbols : $0 \quad S \quad + \quad \cdot$ **Predicate Symbols :** $\equiv$

Logical Symbols : $\bot \quad \wedge \quad \vee \quad \rightarrow$ **Quantifiers :** $\forall \quad \exists$

We don't assume all axioms, but only the following fragment

$$\textbf{Zero addition} : \forall x.\ 0 + x \equiv x$$
$$\textbf{Recursion for addition} : \forall xy.\ (Sx) + y \equiv S(x + y)$$
$$\textbf{Zero multiplication} : \forall x.\ 0 \cdot x \equiv 0$$
$$\textbf{Recursion for multiplication} : \forall xy.\ (Sx) \cdot y \equiv y + x \cdot y$$

$$f : A \rightarrow \text{FA formulas} \quad \text{s.t.} \quad \forall x.\ \ \alpha(x) \ \leftrightarrow \ \beta(f(x))$$

Diophantine constraints

We define expressions containing variables, we call

atomic equations

$$x_i = 1 \quad | \quad x_i + x_j = x_k \quad | \quad x_i \cdot x_j = x_k$$

And evaluations of these expressions for given $\sigma : \mathbb{N} \to \mathbb{N}$

- $[x_i = 1]_\sigma := \sigma(i) = 1$
- $[x_i + x_j = x_k]_\sigma := \sigma(i) + \sigma(j) = \sigma(k)$
- $[x_i \cdot x_j = x_k]_\sigma := \sigma(i) \cdot \sigma(j) = \sigma(k)$

We call a list $L = [e_1, \ldots, e_n]$ of atomic equations e_j a H10 problem and extend $[\ ]_\sigma$ to problems by $[L]_\sigma := [e_1]_\sigma \wedge \ldots \wedge [e_n]_\sigma$.

Satisfiabiliy of diophantine constraints

Given a H10 problem L, we can now ask the question:

Satisfiability

Can L be satisfied? $\leftrightarrow$ Can we show $\exists\sigma.\ [L]_\sigma$?

This question is equivalent to asking if some diophantine equation has a solution. The latter is known to be **undecidable** [Matijasevič, 1970] [Larchey-Wendling and Forster, 2019].

$$f : \text{H10 problems} \to \text{FA formulas} \quad \text{s.t.} \quad \forall L.\ \ \text{sat}(L) \ \leftrightarrow\ \beta(f(L))$$

Embedding H10 problems into FA

Let's look at the following example of an H10 problem

$$L = [x + x = y \ , \ y \cdot y = x]$$

We want to send this to a formula in FA which intuitively expresses the satisfiability of L.

The choice is canonical:

$$\exists x \ \exists y \quad \underbrace{x + x \equiv y \ \wedge \ y \cdot y \equiv x}_{\varepsilon^*(L)}$$

$$\varepsilon : \text{H10 problems} \to \text{FA formulas} \quad \text{s.t.} \quad \forall L.\ \ \text{sat}(L) \ \leftrightarrow \ \beta(\varepsilon(L))$$

Tarski Semantics

We can interpret sentences from our first-order language of arithmetic in the standard model $(\mathbb{N}, 0, S, +, \cdot)$.

Given an environment $\rho : \mathbb{N} \to \mathbb{N}$ we can evaluate terms. We can then use this to define truth of formulas φ in $\mathbb{N}$, for which we write $\mathbb{N} \vDash \varphi$.

Examples

- $\mathbb{N} \vDash (x_1 + x_2 \equiv x_3) \;=\; \forall \rho.\;\; \rho(1) + \rho(2) = \rho(3)$
- $\mathbb{N} \vDash (\forall x.\; 0 + x \equiv x) \;=\; \forall n : \mathbb{N}.\; 0 + n = n$

Tarski Semantics

If we replace $\mathbb{N}$ with some other domain D providing

- $\mathbb{O} : D$
- $\mathbb{S} : D \to D$
- $\oplus : D \times D \to D$
- $\otimes : D \times D \to D$

we get the more general notion of a model $(D, \mathbb{O}, \mathbb{S}, \oplus, \otimes)$ for arithmetic.

Example

$$D \vDash (\forall x.\ 0 + x \equiv 0) \ = \ \forall d : D.\ \mathbb{O} \oplus d = d$$

We call φ valid in FA and write $\text{FA} \vDash \varphi$ iff

$$\forall D \text{ model of FA} \quad \forall \rho.\ \ D \vDash_\rho \varphi$$

$$\varepsilon : \text{H10 problems} \to \text{FA formulas} \quad \text{s.t.} \quad \forall L.\ \ \text{sat}(L) \ \leftrightarrow \ \text{FA} \vDash \varepsilon(L)$$

Canonical Model Homomorphism

If we have some FA model D, we can recursively define a function $\nu : \mathbb{N} \to D$ by

Definition

$$\nu(0) := \mathbb{O} \quad , \quad \nu(x+1) := \nu(x) \oplus \mathbb{S}\ \mathbb{O}$$

Giving us an embedding of $\mathbb{N}$ into any FA model.

By induction over $x : \mathbb{N}$ we can show that ν is a homomorphism:

Homorphism Lemma

$$\nu(x+y) = \nu(x) \oplus \nu(y) \qquad \nu(x \cdot y) = \nu(x) \otimes \nu(y)$$

For the proof of these equations we need the axioms we assumed for FA.

Verification of Reduction

To verify the reduction, we now need to show

Theorem

$$\forall L.\ \mathrm{sat}(L) \ \leftrightarrow\ \mathrm{FA} \vDash \varepsilon(L)$$

Proof.

$\leftarrow$ We use that $\mathbb{N} \vDash \exists^{N} \varepsilon^{*}(L)$. Providing us N elements in $\mathbb{N}$ that give us a solution for L.

$\rightarrow$ By $\mathrm{sat}(L)$ we have a solution σ for L, which we can transport to any model D via the homomorphism ν. $\square$

Closing Remarks

Since the proof works for the fragment FA, it also works for PA. This was very easy to check with Coq.

Advantages of working with Coq

- Definitions can easily be modified; broken proofs will be pointed out
- Admitting proof goals
- Looking up definitions is a matter of seconds
- Standard library with many theorems
- Book-keeping
- Automation

More Work...

I did

- Some results on finite PA models.
- Failed Attempt of an undecidability proof.

In progress right now

- replacing FA $\models$ by FA $\vdash$

Possible next goals

- Tennenbaum's Theorem
- Self-verifying Theories
- Getting PA $\vdash \varphi$ from $\mathbb{N} \models \varphi$

Thank you for your attention!

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