Tennenbaum's Theorem in Constructive Type Theory

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Tennenbaum's Theorem

- Tarski semantics in constr. tpye theory $\rightarrow$ only computable functions,
- in contrast to more classical semantics.
- What about non-standard models?

Tennenbaum's Theorem [Tennenbaum, 1959, Smith, 2014]

There is no recursive non-standard PA model.

Definition

M is a recursive PA model iff there is a bijection $\mu : M \rightarrow \mathbb{N}$ and either addition or multiplication is recursive on the codes $\mu(M)$.

Standard Models

Given any $M \vDash \mathrm{PA}$ we can easily embed the standard model $\mathbb{N}$ by

$$\nu(0) := 0^M \quad ; \quad \nu(n+1) := S^M(\nu(n))$$

we call image points $\overline{n} := \nu(n)$ **standard numbers** / **numerals**.
ν is an inj. homomorphism and we have

$$\nu \text{ surj.} \iff \forall e \underbrace{\exists n.\ e = \nu(n)}_{:= \mathsf{std}(e)} \iff M \cong \mathbb{N}.$$

For now we will assume that $M \ncong \mathbb{N}$ means we have some non-standard element $e : M$.

Ingredients (Textbook Version)

Decidable Divisibility

If M is recursive, then $\overline{n} \mid e$ is decidable for any $n : \mathbb{N}$, $e : M$.

Overspill

If $M \ncong \mathbb{N}$ then for every formula φ with $\forall n : \mathbb{N}.\, M \vDash \varphi(\overline{n})$ there is a non-standard $e : M$ with $M \vDash \varphi(e)$.

Encoding Sets

We can code the finite set $\{3, 8, 14, 21\}$ by

$$c := \pi_3 \cdot \pi_8 \cdot \pi_{14} \cdot \pi_{21} \qquad (\pi_k \text{ is the } k\text{-th prime})$$

Decoding works by uniqueness of prime decomp.
Given any propositially decidable predicate P and bound $n : \mathbb{N}$ we can thus code the finite set $\{u < n \mid P(u)\}$.

Lemma (Coding in $\mathbb{N}$)

If P is prop. decidable then

$$\forall n : \mathbb{N}\ \exists c : \mathbb{N}\ \ \forall u < n.\ P(u) \leftrightarrow \pi_u \mid c$$

$$\text{by Overspill} \qquad \exists c : M\ \ \forall u < e.\ P(u) \leftrightarrow \pi_u \mid c$$

$$\exists c : M\ \ \forall u : \mathbb{N}.\ P(u) \leftrightarrow \pi_u \mid c$$

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Coding

If $M \ncong \mathbb{N}$ and $\lambda n.M \vDash \varphi(\overline{n})$ prop. decidable then there is a code $c : M$ such that $\forall u : \mathbb{N}.\ M \vDash \varphi(\overline{u}) \ \leftrightarrow\ \overline{\pi_u} \mid c$.

Inseparable Formulas

There are disjoint Σ_1-formulas which cannot be seperated by a decidable predicate $D : \mathbb{N} \to \mathbb{P}$.

Proof Sketch

Assume $M \ncong \mathbb{N}$ is recursive.

- Take two disjoint inseparable Σ_1-formulas $\alpha(y) = \exists x.\alpha_0(x, y)$, $\beta(y) = \exists x.\beta_0(x, y)$ where α_0, β_0 are Δ_0.
- Since α, β are disjoint

$$\forall n : \mathbb{N}\ \forall x, y, z < n.\ \neg\ \alpha_0(x, z) \wedge \beta_0(y, z)$$

$$\text{by Overspill} \quad \forall x, y, z < e.\ \neg\ \alpha_0(x, z) \wedge \beta_0(y, z) \qquad (*)$$

  where e is non-standard.
- We define $X := \lambda n. M \vDash \exists x < e.\ \alpha_0(x, \overline{n}) : \mathbb{N} \to \mathbb{P}$.
- X is prop. decidable and can therefore be coded by some $c : M$.
- So it is decidable since $X(n) \leftrightarrow \overline{\pi_n} \mid c$.
- Using $(*)$ and $\neg\mathrm{std}(e)$ we can show that X seperates α, β.
- Contradiction. $\square$

Constructive Type Theory

We will consider the model-theory of PA inside of a constructive type theory (CTT).
So in particular:

- Every model is a type.
- Every function symbol is interpreted as a function in the CTT.

We make two additional (consistent) assumptions:

Markov's Principle (MP)

For every function $f : \mathbb{N} \to \mathbb{B}$ we have $\neg\neg(\exists x. f(x) = 1) \to \exists x. f(x) = 1$.

Church's Thesis (CT)

Every function $\mathbb{N} \to \mathbb{N}$ in the CTT is computable (e.g. μ-recursive).

Tennebaum in CTT

Assuming CT and MP, if $M \vDash \text{PA}$ is a data type then $M \cong \mathbb{N}$.

PA Representibility

How do we use CT? We can combine CT wit a standard result [Smith, 2013]:

For any μ-recursive function $f : \mathbb{N} \to \mathbb{N}$ we have

$$f \text{ is representable in PA} := \exists \varphi_f.\ \Sigma_1\, \varphi_f \ \wedge \\ \forall n : \mathbb{N}.\ \mathsf{PA} \vdash \forall x, \varphi_f(\overline{n}, x) \leftrightarrow \overline{f(n)} = x$$

giving us

$\mathrm{CT}_{\mathrm{PA}}$

Every function $f : \mathbb{N} \to \mathbb{N}$ is representable in PA.

Ingredients in CTT

Decidable Divisibility

If M is recursive, then $\overline{n} \mid e$ is decidable for any $n : \mathbb{N}$, $e : M$.

Overspill

If $M \ncong \mathbb{N}$ then for every formula φ with $\forall n : \mathbb{N}.\, M \vDash \varphi(\overline{n})$ there is a non-standard $e : M$ with $M \vDash \varphi(e)$.

Coding

If $M \ncong \mathbb{N}$ and $\lambda n. M \vDash \varphi(\overline{n})$ prop. decidable then there is a code $c : M$ such that $\forall u : \mathbb{N}.\ M \vDash \varphi(\overline{u}) \ \leftrightarrow\ \overline{\pi_u} \mid c$.

Inseparable Formulas (CT)

There are disjoint Σ_1-formulas which cannot be seperated by a decidable predicate $D : \mathbb{N} \to \mathbb{P}$.

Proof by McCarty

Version of Tennenbaum's Theorem for Heyting arithmetic (HA) due to Charles McCarty [McCarty, 1988, McCarty, 1987].

Tennenbaum for HA

Assuming MP every HA model is isomorphic to $\mathbb{N}$.

For this we change the underlying model-theory to be constructive.

Logical symbols in the PA/HA language will be interpreted as their computational counterparts.

Most notably we have

$$M \vDash \alpha \vee \beta := M \vDash \alpha \ + \ M \vDash \beta$$
$$M \vDash \exists x.\varphi(x) := \Sigma x : M.\ M \vDash \varphi(x)$$

Proof by McCarty

Assume that M has a non-standard element $e : M$ and let α, β the inseparable Σ_1-formulas.

- One can show

$$\mathsf{HA} \vdash \forall x \neg\neg\forall y < x.\ \alpha(y) \vee \neg\alpha(y).$$

- Instantiating the above for e, we get

$$\begin{aligned} & M \vDash \neg\neg\forall y < e.\ \alpha(y) \vee \neg\alpha(y) \\ & M \vDash \quad \forall y < e.\ \alpha(y) \vee \neg\alpha(y) \\ \forall n : \mathbb{N}.\quad & M \vDash \quad \overline{n} < e \to\ \alpha(\overline{n}) \vee \neg\alpha(\overline{n}) \\ \forall n : \mathbb{N}.\quad & M \vDash \quad \alpha(\overline{n}) \vee \neg\alpha(\overline{n}) \end{aligned}$$

- Therefore $\lambda n.M \vDash \alpha(\overline{n})$ is decidable.
- One then checks that $\lambda n.M \vDash \alpha(\overline{n})$ seperates α, β.
- Contradiction. $\square$

Closing Remarks

- McCarty's Proof does not require the model to be countable,
- but we needed to work in HA.
- In the first proof we do not need to retrict to HA.
- In the Coq development right now: $+$ and $\times$ recursive.
- In every proof, we can locate the usage of MP, it is only ever used to show that std is stable.

Do we have something like $\forall \mathsf{PA} \vDash M.\ \mathsf{std}_M$ is stable $\to$ MP?

- Using a different proof approach [Makholm,], we can drop CT, MP and still show $\forall e. \neg\neg \mathsf{std}(e)$.

Bibliography

Makholm, H.
Tennenbaums theorem without overspill.
Mathematics Stack Exchange.
(version: 2014-01-24).

McCarty, C. (1987).
Variations on a thesis: intuitionism and computability.
Notre Dame Journal of Formal Logic, 28(4):536–580.

McCarty, C. (1988).
Constructive validity is nonarithmetic.
The Journal of Symbolic Logic, 53(4):1036–1041.

Smith, P. (2013).
An introduction to Gödel's theorems.
Cambridge University Press.

Smith, P. (2014).
Tennenbaum's theorem.

Tennenbaum, S. (1959).
Non-archimedean models for arithmetic.
Notices of the American Mathematical Society, 6(270):44.

Definability of std

Can we find a formula which picks out the numerals in any model? Well...

Lemma

$M \cong \mathbb{N}$ iff there is a formula φ such that for every $e : M$

$$\mathsf{std}(e) \ \longleftrightarrow \ M \vDash \varphi(e)$$

Proof

If there is such a formula, then

- $M \vDash \varphi(0)$ since $\mathsf{std}(0)$
- $M \vDash \varphi(x) \to \mathsf{std}(x) \to \mathsf{std}(x+1) \to M \vDash \varphi(x+1)$

and hence by induction $\forall x. M \vDash \varphi(x)$ which implies $\forall x. \mathsf{std}(x)$.

Overspill

Can you find a formula which picks out the numerals in a model? Well...

Lemma

$M \ncong \mathbb{N}$ iff for every formula φ

$$\neg\big(\forall e.\ \mathsf{std}(e) \ \leftrightarrow\ M \vDash \varphi(e)\big)$$

Assuming Markov's Principle this implies

Lemma (Overspill)

$M \ncong \mathbb{N}$ then for every formula φ with $\forall n : \mathbb{N}.\ M \vDash \varphi(\overline{n})$ we have

$$\neg\neg\,\exists e.\ \neg\mathsf{std}(e) \ \wedge\ M \vDash \varphi(e)$$

Decidable Divisibility

We can always decide whether $\overline{n} \mid c$ if M is a data type.

Proof

Euclid's Lemma holds in M, so there are unique $d, r : M$ with

$$c = d \otimes \overline{n} \oplus r \text{ with } r < n$$

and since M is a data type we get them computationally. Now we simply check whether $r = 0$ to see if $\overline{n} \mid c$.

Inseparable Sets

Inseparable Sets

There are disjoint enumerable predicates $A, B : \mathbb{N} \to \mathbb{P}$ which are not seperable by a decidable predicate. Meaning, for every dec. $D : \mathbb{N} \to \mathbb{P}$ we can not have

$$\forall n : \mathbb{N}.\ A(n) \to D(n) \text{ and } \forall n : \mathbb{N},\ \neg(D(n) \land B(n))$$

Proof idea

Given an enumeration ψ_n of all formulas, define

$$A(n) := PA \vdash \neg\psi_n(n) \quad \text{and} \quad B(n) := PA \vdash \psi_n(n).$$

Using $\mathsf{CT}_{\mathsf{PA}}$ a decidable predicate D is representable by a Σ_1-formula φ_D. There must be $c : \mathbb{N}$ with $\psi_c = \varphi_D$. By case analysis on $D(c) \lor \neg D(c)$ on now shows that D cannot seperate A and B.

Inseparable Formulas

Inseparable formulas

There are disjoint Σ_1 formulas α, β which are not seperable by a decidable predicate. Meaning, for every dec. $D : \mathbb{N} \to \mathbb{P}$ we can not have

$$\forall n : \mathbb{N}.\ \mathbb{N} \vDash \alpha(n) \to D(n) \ \text{ and } \ \forall n : \mathbb{N},\ \neg(D(n) \land \mathbb{N} \vDash \beta(n))$$

"Disjoint" can mean:

$$\begin{array}{rl} \text{(weak)} & \forall n.\ \mathbb{N} \vDash \neg(\alpha \land \beta)(n) \\ \text{(strong)} & \mathsf{HA} \vdash \neg \exists x. \alpha(x) \land \beta(x) \end{array}$$

We get the weak version by using CT.