The Undecidability of First-Order Logic over Small Signatures

Final Bachelor Talk

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October 14, 2021, Saarland University

First-Order Logic

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- Can we simplify FOL formulas to recover decidability?
  - Restrict signature?
  - Restrict logical connectives?
  - Restrict semantics?
- Semantics: what does "truth" mean?
  - Satisfaction in all models
  - Satisfaction in finite models
  - Abstract deduction system

(Un)decidable fragments of FOL

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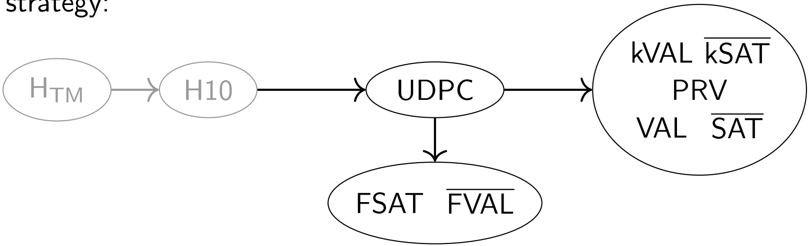
Reduction strategy:



Uniform Diophantine Pair Constraints

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Characterizing properties

- $(a,0) \wr (a+1,0)$
- $(a,b'+1) \wr (c'+1,d)$ iff $(a,b') \wr (c',d') \land (d',b') \wr (d,d')$

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- UDPC and $\wr$ originally due to Andrej Dudenhefner

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$\Rightarrow$ VAL is undecidable for binary signature

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  - classical provability PRV_c similarly undecidable, assuming LEM

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  - Now: Given model encoding solution, extract it

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Idea: Still reduce from UDPC

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- There is no sound, complete, enumerable deduction system for finite semantics
  - FVAL is co-enumerable and undecidable

Summary

We have shown undecidability of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment

Summary

We have shown undecidability of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment
- Note: We defined undecidable $P :=$ decidable $P \rightarrow$ semidecidable $\overline{\mathsf{H}_{\mathsf{TM}}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \perp$-fragment

Coq mechanization:

- $\sim$900 LoC for PRV and corollaries
  - [Kirst and Hermes, 2021]: 4.5k LoC

[2] defined by semidecidability of $\overline{\mathsf{H}_{\mathsf{TM}}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment

Coq mechanization:

- $\sim$900 LoC for PRV and corollaries
- $\sim$1200 LoC for FSAT, FVAL
  - [Kirst and Larchey-Wendling, 2020]: >5k LoC

[2] defined by semidecidability of $\overline{\mathsf{H}_{\mathsf{TM}}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment

Coq mechanization:

- $\sim$900 LoC for PRV and corollaries
- $\sim$1200 LoC for FSAT, FVAL
- $\sim$200 LoC for UDPC
  - Requires undecidability of H10 [Larchey-Wendling and Forster, 2019]: 8k LoC

[2] defined by semidecidability of $\overline{\mathsf{H}_{\mathsf{TM}}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment

Coq mechanization:

- $\sim$900 LoC for PRV and corollaries
- $\sim$1200 LoC for FSAT, FVAL
- $\sim$200 LoC for UDPC

Pain points:

- Working with de Brujin indices and double negation is unintuitive

[2] defined by semidecidability of $\overline{\mathrm{H}_{\mathrm{TM}}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment

Coq mechanization:

- $\sim$900 LoC for PRV and corollaries
- $\sim$1200 LoC for FSAT, FVAL
- $\sim$200 LoC for UDPC

Pain points:

- Working with de Brujin indices and double negation is unintuitive
- Constructing abstract deduction system proofs is painful
  - Existing FOL toolbox [Hostert et al., 2021] could be expanded

[2] defined by semidecidability of $\overline{\mathsf{H}_{\mathsf{TM}}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \perp$-fragment

Coq mechanization:

- ~900 LoC for PRV and corollaries
- ~1200 LoC for FSAT, FVAL
- ~200 LoC for UDPC

Pain points:

- Working with de Brujin indices and double negation is unintuitive
- Constructing abstract deduction system proofs is painful

Future work:

- FVAL for $\forall, \rightarrow$-fragment

[2] defined by semidecidability of $\overline{H_{TM}}$

Summary

We have shown undecidability[2] of

- PRV, VAL, kVAL for binary signature over $\forall, \rightarrow$-fragment
- SAT, kSAT, FSAT, FVAL for binary signature over $\forall, \rightarrow, \bot$-fragment

Coq mechanization:

- $\sim$900 LoC for PRV and corollaries
- $\sim$1200 LoC for FSAT, FVAL
- $\sim$200 LoC for UDPC

Pain points:

- Working with de Brujin indices and double negation is unintuitive
- Constructing abstract deduction system proofs is painful

Future work:

- FVAL for $\forall, \rightarrow$-fragment
- PRV_c with MP

[2] defined by semidecidability of $\overline{\mathsf{H}_{\mathsf{TM}}}$

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