Mechanized Undecidability of Halting Problems for Reversible Machines

Final Talk

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Outline

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- See (Bennett, 2003) for a more thorough treatment on Landauer's Principle.

Goal

The goal of this thesis is to mechanize in Coq the (un)-decidability of halting problems for:

- Reversible FRACTRANs
- Reversible 2-counter machines
- Weakly-reversible 2-dimensional cellular automata

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A machine M is *weakly reversible* iff for all its configurations s, t, and u, if M steps from s to u and from t to u and u is not halting then $s = t$.

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Theorem

Reversible FRACTRAN programs have decidable halting problems.

Proof.

The key observation here is that reversible FRACTRAN programs essentially contain at most one fraction[1], for which one can construct halting deciders for. □

[1] with insights from a private communication with Dominique Larchey-Wendling

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- A counter machine is a list of instructions (from some instruction set) whose configurations are pairs $(i, \overline{v})$ of addresses i and counters values $\overline{v}$.

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- For MM, DEC $x\ j$ at address i jumps to j if the counter x contains zero (leaving it unchanged), otherwise decrements the counter x and jumps to $i + 1$.
- For MMA/CM, DEC $x\ j$ at address i decrements the counter x and jumps to j if it contains a positive number, otherwise it leaves x unchanged and jumps to $i + 1$.

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- Hence we use Morita's counter machines (Morita, 1996).

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- MM and MMA/CMs are deterministic by constructions: only one instruction per address.
- But Morita's counter machines can be non-deterministic.

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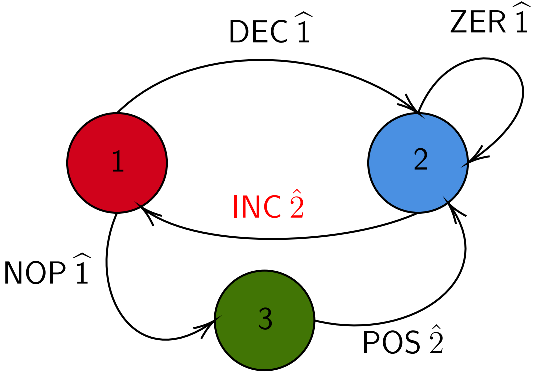

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Need to characterize reversible Morita's counter machines syntatically.

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- A Morita's counter machine is *intensionally reversible* if none of its instructions overlap in range.

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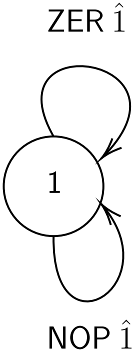

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We partially mechanize Morita's construction by using graph representations.

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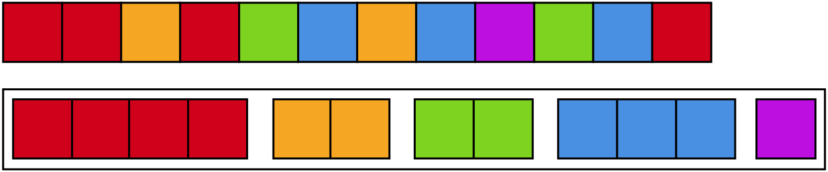

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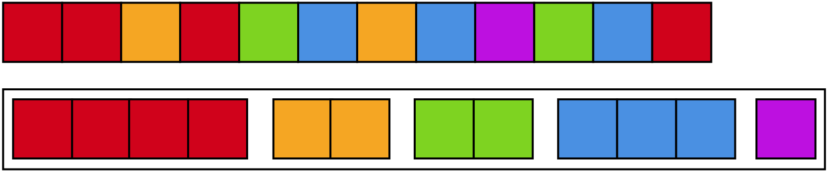

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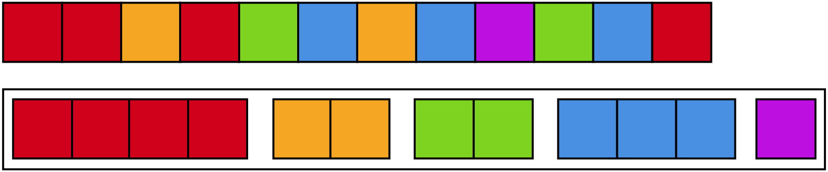

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Each step of Morita's construction can then be implemented as a simple map or flat-map.

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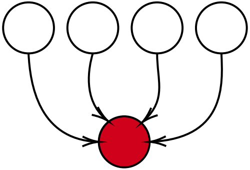

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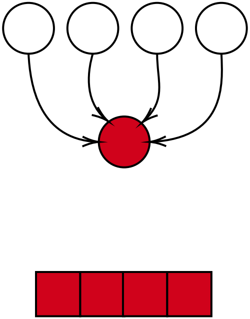

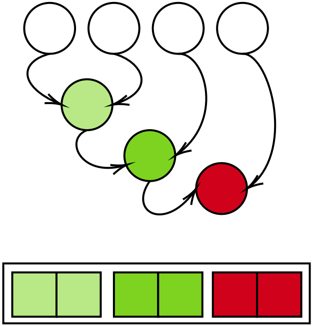

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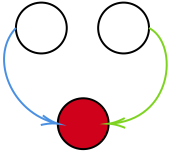

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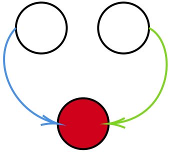

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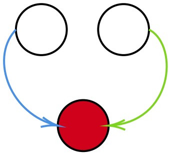

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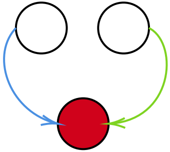

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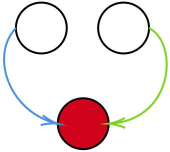

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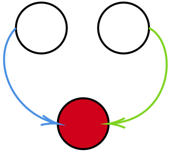

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- Need a way to construct reversible loops.

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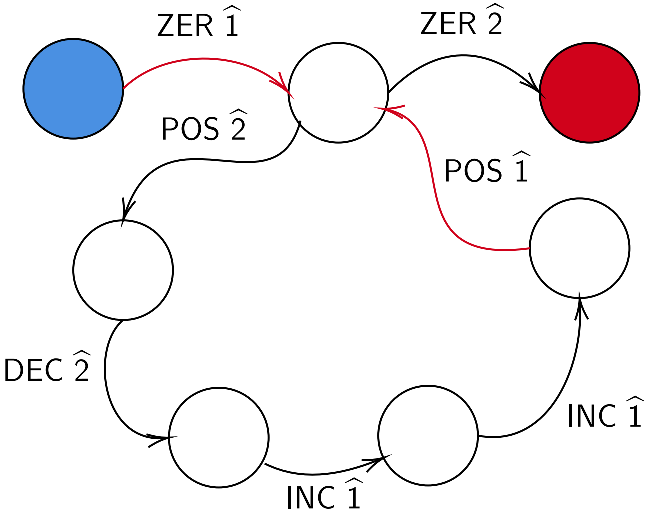

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- Thus incrementing the first counter becomes multiplication by 2, for example.
- The crucial point here is to preserve reversibility, but we already know how to construct loops that preserve reversibility.
- We did not mechanize this step due to time constraint.

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- The local update rule is applied homogenously, globally.

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- Its configurations are functions $s : \mathbb{Z} \to \Sigma$, which can be thought of as arrays of cells.

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Rule 110

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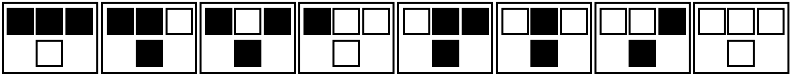

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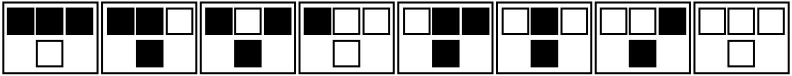

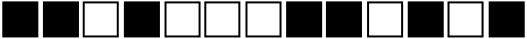

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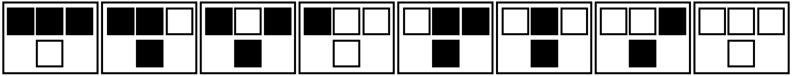

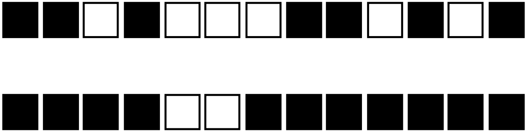

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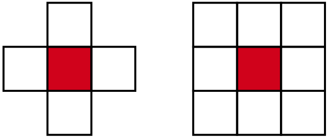

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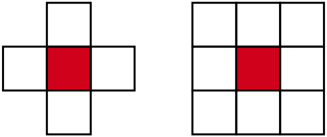

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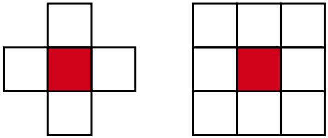

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- Its configurations are functions $s : \mathbb{Z} \to \mathbb{Z} \to \Sigma$.
- Similar to CA1s, there are quiescent letters and spatially-finite configurations.

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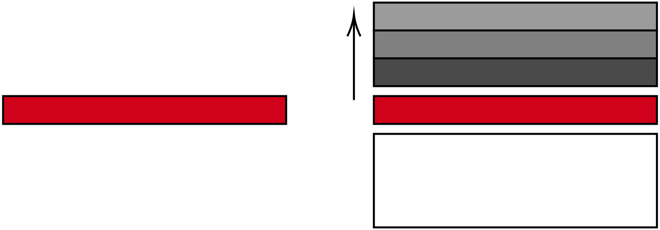

Outline

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- Cellular automata can have almost arbitrary control flow mechanisms.

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- Morita's counter machines, viewed as lists, do not provide enough structure to implement Morita's construction. Our graph representation significantly simplifies our mechanization of Morita's construction.
- The old version of binary Turing machine in the library contains too many edge cases. The new version of binary Turing machine[2] in the library was partly influenced by our discussion on reduction to CA1.

[2] https://github.com/uds-psl/coq-library-undecidability/pull/143

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- Reduction from CA1 to weakly-reversible CA2.

Thank you for your attention!

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- Intensional determinism is also sound.

Deterministic simulation

Let $\Rightarrow_1$ and $\Rightarrow_2$ be step relations. Assuming the following hold:

- For all configurations s_1, s_2, and t_1, if $s_1 \Rightarrow t_1$ and sync s_1 s_2 then there exists t_2 such that $s_2 \Rightarrow_2^+ t_2$ and sync t_1 t_2.
- For all configurations s_1 and s_2, if $\Rightarrow_1$ is stuck at s_1 and sync s_1 s_2 then $\Rightarrow_2$ terminates starting from s_2.
- $\Rightarrow_1$ is decidable.
- $\Rightarrow_2$ is deterministic.

then we have that for all configurations s_1 and s_2 that are in sync, $\Rightarrow_1$ terminates starting from s_1 iff $\Rightarrow_2$ terminates starting from s_2.