Formal Construction of Set-Theoretic Models for an Extended Calculus of Constructions

joint work with Chad E. Brown

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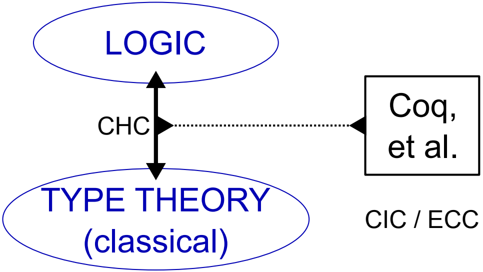
Outline

LOGIC

TYPE THEORY
(classical)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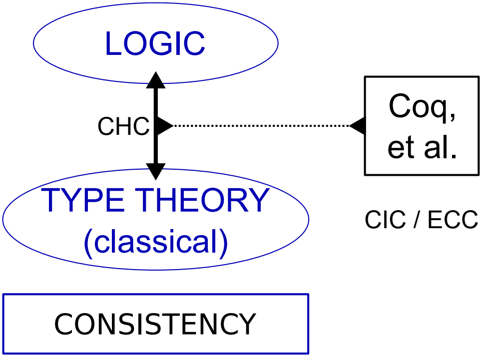
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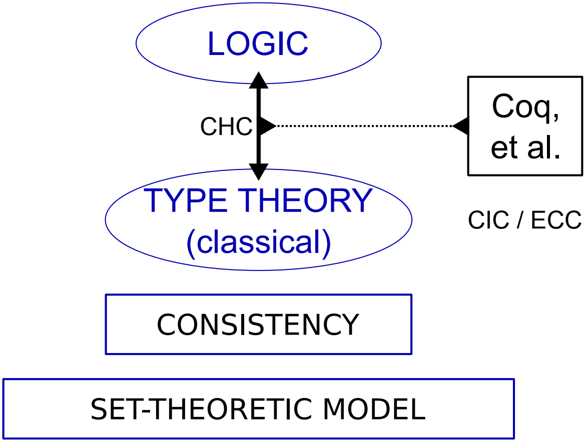
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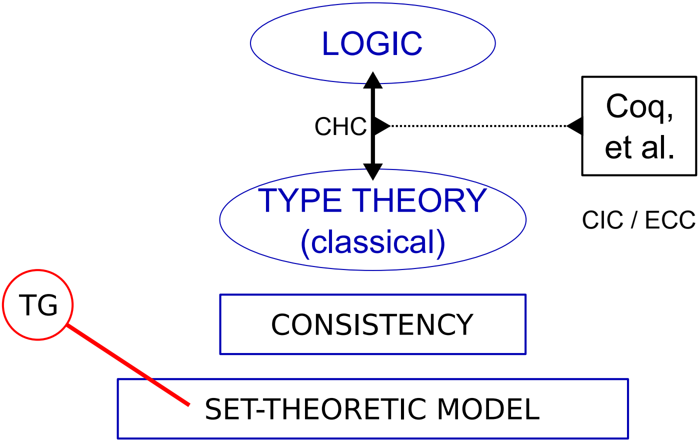
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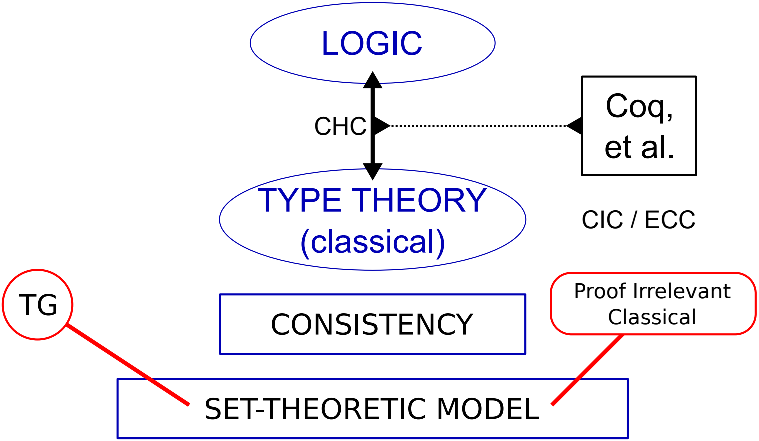
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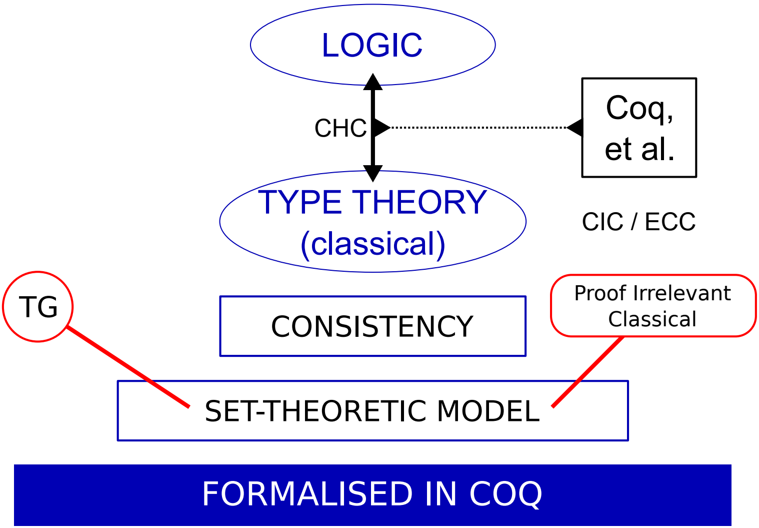
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FORMALISED IN COQ

Luo's Extended Calculus of Constructions [4]

Term structure

- The kinds Prop and $\text{Type}_0, \text{Type}_1, \text{Type}_2, \ldots$ are terms
- Variables $(x, y, \ldots)$ are terms
- Let M, N, A and B be terms, then

$$\Pi x : A,\ B \,|\, \lambda x : A.\, N \,|\, M\, N \,|$$
$$\Sigma x : A,\ B \,|\, \mathbf{pair}_{\Sigma x:A,\, B}(M, N) \,|\, \pi_1(M) \,|\, \pi_2(M)$$

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  are terms

Properties

- Strongly normalising
- No strong sums in Prop (would lead to inconsistency)
- Kinds are (fully) cumulative:

$$\text{Prop} \preceq \text{Type}_0$$
$$\text{Type}_n \preceq \text{Type}_{n+1}$$

Tarski-Grothendieck set theory: ZFC & GU

$$
\begin{aligned}
&\forall x,\ x \notin \emptyset \\
x \in \{a, b\} &\iff x = a \lor x = b \\
x \in \bigcup A &\iff \exists X \in A,\ x \in X \\
y \in \{F\,x \mid x \in X\} &\iff \exists z,\ z \in X \land y = F\,z \\
X \in \mathscr{P}(A) &\iff X \subseteq A \\
X = Y &\iff X \subseteq Y \land Y \subseteq X \\
(\forall X,\ (\forall x \in X,\ P\,x) &\to P\,X) \to \forall X,\ P\,X
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Choice: should follow from ε in meta theory

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Grothendieck Universes

- Transitive set ($X \in U,\ x \in X \implies x \in U$)
- Closed under above operators (e.g. $X \in U \implies \mathscr{P}(X) \in U$)
- For every X there is a *least* universe $U := \mathrm{G}_X$ such that $X \in \mathrm{G}_X$
- Implies infinity ($\mathrm{G}_\emptyset$ is inf.)

Model Construction

Abstract Model

- Some generic set constructions:
  singletons, indexed unions, separation, ordered pairs

Concrete Model

Model Construction

Abstract Model

- Some generic set constructions:
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- Specific set constructions to reflect parts of ECC:

$$[\![\mathrm{Prop}]\!] := 2$$
$$[\![\mathrm{Type}_0]\!] := \mathrm{G}_\emptyset$$

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- State soundness …

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Concrete Model

- Formalise syntax, environments, typing rules
- PTS-style or JE conversion formulation
- State soundness …
- …and prove it?

Soundness

- If $\Gamma \vdash M : A$ is derivable, then it is valid in the model:

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- We have to show $[\![M\ N]\!]_\gamma \in [\![B\,[x := N]]\!]_\gamma$

Soundness, JE vs. PTS

$$(\textsc{conv})\ \frac{\Gamma \vdash_{\mathrm{JE}} M : A \qquad \Gamma \vdash_{\mathrm{JE}} A = B : \mathrm{Type}_i}{\Gamma \vdash_{\mathrm{JE}} M : B}$$

$$(\textsc{conv})\ \frac{\Gamma \vdash_{\mathrm{PTS}} M : A \qquad \Gamma \vdash_{\mathrm{PTS}} B : \mathrm{Type}_i}{\Gamma \vdash_{\mathrm{PTS}} M : B}\ A \approx B$$

The JE case?

The PTS case?

Are they equivalent?

Soundness, JE vs. PTS

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The JE case?

Solved (e.g. Barras [2], Lee & Werner [3]).

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The JE case?

Solved (e.g. Barras [2], Lee & Werner [3]).

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Are they equivalent?

Unknown in the general case, approximations exist.

Results so Far . . .

- Adams [1]: equivalent, given uniqueness of types

Results so Far …

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Results so Far . . .

- Adams [1]: equivalent, given uniqueness of types
- Miquel & Werner [5]: circumvent problems in PTS case with syntactic annotations to ensure well-sortedness
- Pagano, Coquand et al. (02/2012): equivalent, when dropping impredicativity (norm. by eval.)

Model Properties of Interest

Consistency

- A statement is consistent when we can exhibit a satisfying model
- We construct a proof-irrelevant, classical model
- PI, DN, PE, FE, XM, … should be satisfied in the model

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Independence

- A statement is independent when it is consistent but not provable
- To refute provability, exhibit a non-satisfying model
- E.g. existence of an infinite type in Type_0, formally

$$\exists X : \mathrm{Type}_0,\ \exists f : X \to X, (\exists x : X,\ \forall y : X,\ f y \neq x) \wedge (\forall y\, z : X,\ f y = f z \to y = z)$$

Mutual Consistency of Standard Library

Mutual Consistency

- $(\Gamma,\ A\ \textbf{consistent}) \land (\Gamma,\ B\ \textbf{consistent}) \nRightarrow (\Gamma,\ A,\ B\ \textbf{consistent})$
- XM and ¬PI are both *separately* consistent with CiC …
- … but {XM, ¬PI} is inconsistent with CiC

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ECC representation of CiC axioms

- Consider $P : \mathrm{Prop}_{\mathrm{CiC}}$, find suitable $Q : \mathrm{Prop}_{\mathrm{ECC}}$
- such that $\vdash_{\mathrm{CiC}} Q \leftrightarrow P$ (write $Q \overset{\mathrm{ECC}}{\multimapdotboth} P$)

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Example: Proof Irrelevance

- In Coq / CiC: `forall (P:Prop) (p1 p2:P), p1 = p2.`
- In ECC: $\Pi P : \mathrm{Prop},\ \Pi u : P,\ \Pi v : P,\ u =_P v$
- Where $u =_P v := \Pi R : P \to \mathrm{Prop},\ R\ u \to R\ v$
- The abstract version of $u =_P v$ an Coq's = provably coincide.

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Axioms in the Library

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Classical classical, functional_extensionality, Extensionality_Ensembles

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PI proof_irrelevance, eq_rect_eq, JMeq_eq

Mutual Consistency of Standard Library

Axioms in the Library

- **Classical**: classical, functional_extensionality, Extensionality_Ensembles
- **Choice**: epsilon_statement, constructive_(in)definite_description, dependent_unique_choice, relational_choice
- **PI**: proof_irrelevance, eq_rect_eq, JMeq_eq
- **Reals**: archimed, completeness, Rplus_assoc, …

Thesis Aims

For my thesis I want to . . .

- complete two abstract models
- formalise ECC concretely with JE conversion
- proof soundness for this scenario
- and show mutual consistency of the standard library

Given spare time . . .

- I'd like to investigate the PTS vs. JE problem

References

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Pure type systems with judgemental equality.
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Logical Methods in Computer Science, 7(4), 2011.

Zhaohui Luo.
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Alexandre Miquel and Benjamin Werner.
The Not So Simple Proof-Irrelevant Model of CC.
In *TYPES*, pages 240–258, 2002.

Appendix

Consistency

- Certain types are not inhabited
- A **provable** $::= \exists \mathscr{D}, \vdash \mathscr{D} : A$
- Γ **consistent** $::= \neg\exists \mathscr{D}, \Gamma \vdash \mathscr{D} : \bot$
- We take $\bot : \mathrm{Prop} := \forall P : \mathrm{Prop},\ P$
- ECC is consistent (Proof via Strong Normalisation)
- XM, PE, FE, PI are consistent additions to ECC & CiC
- Set-theoretic Models: $[\![A]\!] \neq \emptyset$

Mutual Consistency & Independence

Independence

- A **consistent** $[\nvdash \neg A]$, not (A **provable**) $[\nvdash A]$
- To refute provability: provide a model where $[\![A]\!] = \emptyset$
- XM is independent from ECC
- (Type_0 contains inf. types, like $\mathbb{N}$) is independent from ECC

Mutual Consistency

- $(\Gamma,\ A$ **consistent**$) \wedge (\Gamma,\ B$ **consistent**$) \nRightarrow (\Gamma,\ A,\ B$ **consistent**$)$
- XM and ¬PI are both *separately* consistent with CiC ...
- ... but {XM, ¬PI} is inconsistent with CiC

Constructions in TG & Meta Theory

- Singleton Sets: $\{x\}$
- 1: $\{\emptyset\} = \mathscr{P}(\emptyset)$
- 2: $\{\emptyset, 1\} = \mathscr{P}(1)$
- Indexed Union: $\bigcup_{i \in I} X_i$
- Separation: $\{x \in X \mid Px\}$
- Ordered Pairs (Kuratowski): $(a, b)_{\mathrm{k}}$
- Cartesian Product: $A \times B$

Related Lemmas

Introduction and elimination rules, correctness statements and useful equalities with respect to the special sets 0, 1 and 2.

Meta Theory

We use classical CiC with extensionality principles and Hilbert's ε.

The ECC Model

Kinds:

$$
\begin{aligned}
[\![\mathrm{Prop}]\!] &:= 2 \\
[\![\mathrm{Type}_0]\!] &:= \mathrm{G}_\emptyset
\end{aligned}
$$

Functions (using Aczel's encoding):

$$
\begin{aligned}
\mathtt{ap}\ f\ x &:= \{y \mid (x, y) \in f\} \\
\mathtt{lam}\ d\ F &:= \{(x, y) \mid x \in d \wedge y \in F\ x\} \\
\mathtt{Pi}\ d\ Y &:= \{\mathtt{lam}\ d\ F \mid \forall x \in d,\ F\ x \in Y\ x\}
\end{aligned}
$$

Strong Sums & Pairs:

- $\mathtt{Sig}\ d\ Y := \mathtt{lam}\ d\ Y$
- Pairs: $(a, b) := \{\{a\}, \{a, b\}\}$

Preliminary Results

True and False

- Both are in $[\![\text{Prop}]\!]$
- $[\![\text{FALSE}]\!] = 0$
- $[\![\text{TRUE}]\!] = 1$

Leibniz Equality

- defined in object logic, one for each type level
- coincides with Coq's equality on our meta type `set`
- asserts that domains are ok

Proof Irrelevance

- We have shown $[\![\text{PI}]\!] = 1$
- i.e. inhabited ...
- ... and in $[\![\text{Prop}]\!]$

Barras [2]

- Defines signatures for CC and CC_ω models
- Works in an intuitionistic setting
- Models are proof-irrelevant
- Implements his signatures using HF and IZF
- Proves soundness of his signatures when using JE
- Obtains soundness for CC with Conversion via Adams
- Won't work for CC_ω since we lack *uniqueness of types*
- Fully formalised in Coq
- Models for our signatures also satisfy his signatures.

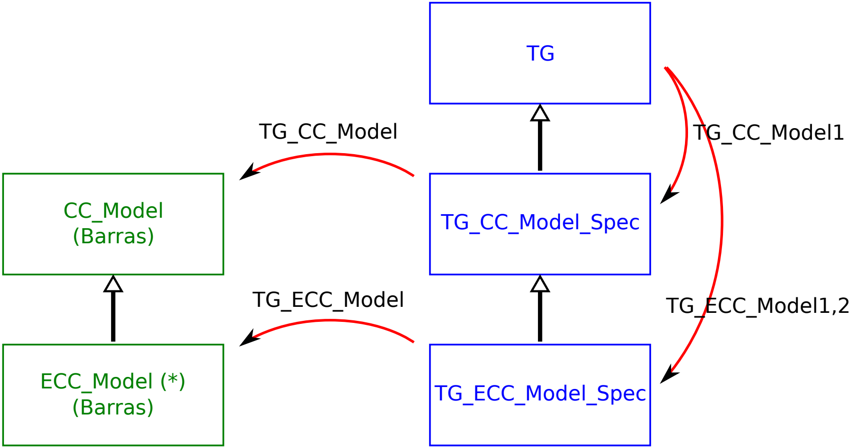
Werner, Lee & Miquel [5, 3]

- Initially compared proof theoretic strength of CiC and ZFC
- Models are proof-irrelevant
- Later mostly focused on the soundness problem.
- 'Solved' by syntactically annotating variables with sorts and dropping $\mathrm{Prop} \leq \mathrm{Type}_0$
- Partially formalised in Coq
- They aim for CiC but exclude Inductive Propositions

Remarks on the Meta theory

- We work in Coq: CiC
- Add `ClassicalFacts`: relates XM, PD, PE, PI, …
- Add `Classical_Prop`: XM, DN, Peirce, PI, …
- Add `FunctionalExtensionality`: FE, …
- Add `Epsilon`: Hilbert's ε, Church's ι
- CDP: $\forall P : \mathrm{Prop},\ P + (\neg P)$ follows from DN and ε.
- $\forall P : \mathrm{Type},\ P + (P \to \bot)$ follows from CDP and ε.

Barras' Framework



Infinite Types in Type_0

$$\exists X : \mathrm{Type}_0,\ \exists f : X \to X, (\exists x : X,\ \forall y : X,\ fy \neq x) \wedge$$
$$(\forall y\, z : X,\ fy = fz \to y = z)$$

f is a function on X which is *injective* but *not surjective*.
This implies that X is infinite.

- Not satisfied in the $[\![\mathrm{Type}_0]\!] := \mathrm{G}_\emptyset$ model;
  (any injective f on X is also surjective – classical)
- Satisfied in the $[\![\mathrm{Type}_0]\!] := \mathrm{G}_{\mathrm{G}_\emptyset}$ model;
  Use $X := \mathrm{G}_\emptyset$ and $f := \mathscr{P}$

Encoding Functions

What's wrong with the standard graph encoding of functions?

- The function space $\top \to \top$ contains exactly one element, the function mapping $\emptyset$ to $\emptyset$.
- with standard graph-encoding: $[\![\top \to \top]\!] = \{\{(\emptyset, \emptyset)\}\} \notin 2$
- however, we want $[\![\top \to \top]\!] = 1 \in 2$
- but $\emptyset \neq \{(\emptyset, \emptyset)\}$!
- with the alternative function encoding, the two sides match up.