Master's thesis - first talk:

Tableaux for Higher-Order Logic with If-Then-Else, Description and Choice

by Julian Backes on June 26, 2009

Advisor: Dr. Chad E. Brown
Supervisor: Prof. Dr. Gert Smolka

Contents

- Basics
- Previous work: Fragments of higher order logic
- Extending the fragments
  - if-then-else
  - Description and Choice

Basics

Simply typed higher order logic and tableaux

Photo: kommunikationserfolg.com

Basics: Syntax/Semantics

- Context: Simply typed higher order logic
- Syntax:
  - Types (σ, τ, μ): $\tau ::= \iota \mid o \mid \tau\ \tau$
  - Terms (s, t, u, v): $t ::= x \mid c \mid tt \mid \lambda x.t$
  - Logical constants: $\neg$, $\wedge$, $\vee$, $\forall_\tau$, $\exists_\tau$, $=_\tau$, $\longrightarrow$, $\top$, $\bot$
- Typed terms as usual, we only consider well-typed terms
- Semantics:
  - o boolean sort, containing 1/true/top/$\top$ and 0/false/bottom/$\bot$
  - ι non-empty set of individuals
  - τ set of all total functions (standard interpretation) or subset of all total total functions (Henkin/non standard interpretation)

Basics: Tableau systems

- General idea: Proof by contradiction
- Instead of proving the validity of a formula, we show that the negation of the formula is unsatisfiable / refutable / yields $\bot$
- Tableau rules:

$$\frac{A}{A_1 \mid \ldots \mid A_n}\; A \subsetneq A_i \qquad\qquad \textsc{Closed}\;\frac{A}{\bot}$$

- For simplicity: We only write what is needed in A to apply a rule and what is added in the A_i

Fragments of higher order logic

Basic, EFO, Full

Fragments of higher order logic

- Chad E. Brown, Gert Smolka: "Terminating Tableaux for the Basic Fragment of Simple Type Theory" (Basic)
  - No λ, only $\bot$, $\neg$, $\wedge$, $=_\tau$ as logical constants
  - No higher-order equations, only higher order disequations
  - Tableau system is complete wrt standard models, cut-free, terminating
- Chad E. Brown, Gert Smolka: "Extended First-Order Logic" (EFO)
  - Supports λ, $\bot$, $\neg$, $\wedge$, $=_\tau$, $\forall_\iota$ as logical constants, only higher order disequations
  - Tableau system is complete wrt standard models, cut-free, not terminating

Fragments of higher order logic ctd.

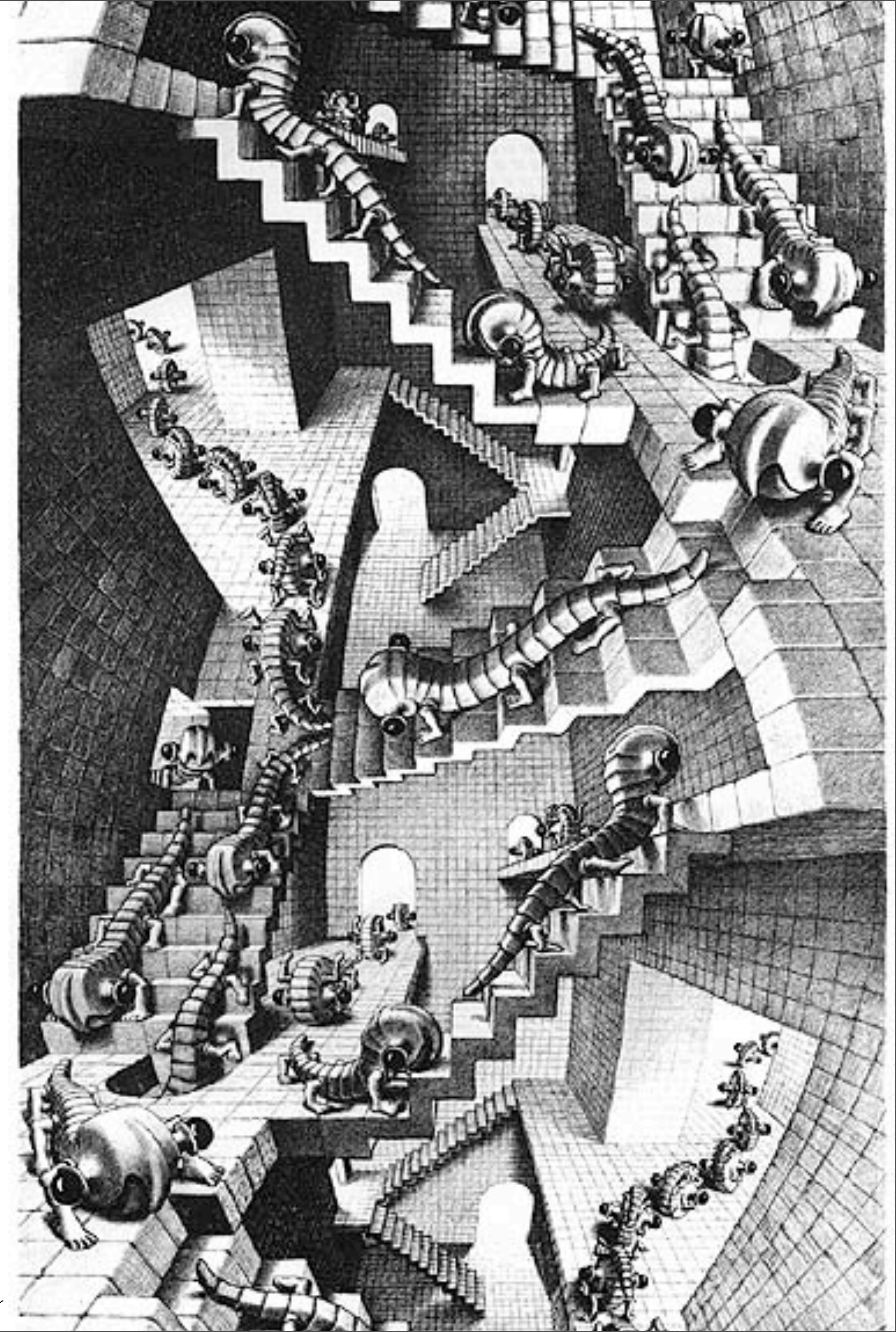
- Chad E. Brown, Gert Smolka: "Complete Cut-Free Tableaux for Equational Simple Type Theory" (Full)
  - Full higher order logic, i.e. supports λ and higher order equations
  - Complete wrt Henkin/non-standard models, cut-free, not terminating
- Some tableau rules from the three fragments:

$$\text{MAT}\ \frac{xs_1 \dots s_n, \neg x t_1 \dots t_n}{s_1 \neq t_1 \mid \dots \mid s_n \neq t_n} \qquad \forall\ \frac{\forall_\iota s}{[st]}\ t : \iota \qquad \text{FE}\ \frac{s \neq_{\sigma\tau} t}{[sx] \neq [tx]}\ x : \sigma \text{ fresh}$$

$$\text{CON}\ \frac{s =_\iota t, u \neq_\iota v}{s \neq u, t \neq u \mid s \neq v, t \neq v} \qquad \text{FQ}\ \frac{s =_{\sigma\tau} t}{[su] = [tu]}\ u : \sigma$$

Extending the fragments

If-Then-Else, Description, Choice



Drawing: M. C. Escher

Extending the fragments

- Our (or my) goal: Extending the presented fragments by adding new logical constants
- Two conditions:
  - The new logical constants should make the fragment more powerful; adding $\neg$ if we already have $\longrightarrow$ and $\bot$ does not bring more power
  - Existing properties like cut-freeness, completeness or termination must be preserved

If-Then-Else

- Let $if_\tau : o\tau\tau\tau$ be the logical constant interpreted as if-then-else
    - Example 1: $if_N \perp 15\ 20 = 20$
    - Example 2: $(if_{NN} \top (\lambda x.7\text{-}x)\ (\lambda x.15\text{-}x))\ 3 = (\lambda x.7\text{-}x)\ 3 = 4$

If-Then-Else

- Let $if_\tau : o\tau\tau\tau$ be the logical constant interpreted as if-then-else
  - Example 1: $if_N \perp 15\ 20 = 20$
  - Example 2: $(if_{NN} \top (\lambda x.7\text{-}x)\ (\lambda x.15\text{-}x))\ 3 = (\lambda x.7\text{-}x)\ 3 = 4$
- Main difference to other logical constants:
  - if_τ does not (always) return something of type o, e.g. if_ι
  - => if_τ does not (always) occur as the "head" in a formula

If-Then-Else

- Let $if_\tau : o\tau\tau\tau$ be the logical constant interpreted as if-then-else
    - Example 1: $if_N \perp 15\ 20 = 20$
    - Example 2: $(if_{NN} \top (\lambda x.7\text{-}x)\ (\lambda x.15\text{-}x))\ 3 = (\lambda x.7\text{-}x)\ 3 = 4$
- Main difference to other logical constants:
    - if_τ does not (always) return something of type o, e.g. if_ι
    - => if_τ does not (always) occur as the "head" in a formula
- Let's have a look at some tableau rules...

Tableau rules for If-Then-Else

$$\text{IF}_\iota \; \frac{v' \neq_\iota (\text{if}\, s\, t\, u)\, v_1 \ldots v_n}{s, t\, v_1 \ldots v_n \neq_\iota v' \;\mid\; \neg s, u\, v_1 \ldots v_n \neq_\iota v'} \; n \geq 0$$

$$\text{IF}_\iota \; \frac{(\text{if}\, s\, t\, u)\, v_1 \ldots v_n \neq_\iota v'}{s, t\, v_1 \ldots v_n \neq_\iota v' \;\mid\; \neg s, u\, v_1 \ldots v_n \neq_\iota v'} \; n \geq 0$$

Tableau rules for If-Then-Else

$$\text{IF}_\iota \ \frac{v' \neq_\iota (\text{if}\, s\, t\, u)\, v_1 \ldots v_n}{s, t\, v_1 \ldots v_n \neq_\iota v' \ \mid \ \neg s, u\, v_1 \ldots v_n \neq_\iota v'} \ n \geq 0$$

$$\text{IF}_\iota \ \frac{(\text{if}\, s\, t\, u)\, v_1 \ldots v_n \neq_\iota v'}{s, t\, v_1 \ldots v_n \neq_\iota v' \ \mid \ \neg s, u\, v_1 \ldots v_n \neq_\iota v'} \ n \geq 0$$

$$\text{IF}_o \ \frac{(\text{if}\, s\, t\, u)\, v_1 \ldots v_n}{s, t\, v_1 \ldots v_n \ \mid \ \neg s, u\, v_1 \ldots v_n} \ n \geq 0$$

$$\text{IF}_\neg \ \frac{\neg((\text{if}\, s\, t\, u)\, v_1 \ldots v_n)}{s, \neg(t\, v_1 \ldots v_n) \ \mid \ \neg s, \neg(u\, v_1 \ldots v_n)} \ n \geq 0$$

Tableau rules for If-Then-Else

$$\mathrm{IF}_\iota \ \frac{v' \neq_\iota (\mathrm{if}\, s\, t\, u)\, v_1 \ldots v_n}{s, t\, v_1 \ldots v_n \neq_\iota v' \ \mid\ \neg s, u\, v_1 \ldots v_n \neq_\iota v'} \ n \geq 0$$

$$\mathrm{IF}_\iota \ \frac{(\mathrm{if}\, s\, t\, u)\, v_1 \ldots v_n \neq_\iota v'}{s, t\, v_1 \ldots v_n \neq_\iota v' \ \mid\ \neg s, u\, v_1 \ldots v_n \neq_\iota v'} \ n \geq 0$$

$$\mathrm{IF}_o \ \frac{(\mathrm{if}\, s\, t\, u)\, v_1 \ldots v_n}{s, t\, v_1 \ldots v_n \ \mid\ \neg s, u\, v_1 \ldots v_n} \ n \geq 0$$

$$\mathrm{IF}_\neg \ \frac{\neg((\mathrm{if}\, s\, t\, u)\, v_1 \ldots v_n)}{s, \neg(t\, v_1 \ldots v_n) \ \mid\ \neg s, \neg(u\, v_1 \ldots v_n)} \ n \geq 0$$

- With these tableau rules, I already proved completeness wrt EFO formulas with if_τ
- Moreover, if_τ can also be added to the other two fragments while preserving completeness, cut-freeness and (non) termination

Description and Choice

- Description $D_\tau : (\tau o)\tau$ is defined as

$$\forall p.(\exists! x.px) \to p(Dp)$$

- Choice $C_\tau : (\tau o)\tau$ is defined as

$$\forall p.(\exists x.px) \equiv p(Cp)$$

Description and Choice

- Description $D_\tau : (\tau o)\tau$ is defined as

$$\forall p.(\exists! x.px) \to p(Dp)$$

- Choice $C_\tau : (\tau o)\tau$ is defined as

$$\forall p.(\exists x.px) \equiv p(Cp)$$

- Main difference to other logical constants (including if_τ):
  - The interpretation of C and D is not unique
  - Example for C: If a subset contains more than one element, we do not know which element will be chosen by C

Tableau rules for Choice

- G. Mints: "Cut-Elimination for Simple Type Theory with an Axiom of Choice"; Journal of Symbolic Logic 64 (2), 479-485. 1999. (the paper is from 1996)
- Does not use λ-calculus but "Takeuti's style" / relational style (see next slide)
- Uses a proof system based on a sequent calculus
- Author proves completeness with Choice but without Cut

Tableau rules for Choice

- G. Mints: "Cut-Elimination for Simple Type Theory with an Axiom of Choice"; Journal of Symbolic Logic 64 (2), 479-485. 1999. (the paper is from 1996)
- Does not use λ-calculus but "Takeuti's style" / relational style (see next slide)
- Uses a proof system based on a sequent calculus
- Author proves completeness with Choice but without Cut
- Assume C p, C p' occur as subterms on the branch

Tableau rules for Choice

- G. Mints: "Cut-Elimination for Simple Type Theory with an Axiom of Choice"; Journal of Symbolic Logic 64 (2), 479-485. 1999. (the paper is from 1996)
- Does not use λ-calculus but "Takeuti's style" / relational style (see next slide)
- Uses a proof system based on a sequent calculus
- Author proves completeness with Choice but without Cut
- Assume C p, C p' occur as subterms on the branch

$$\textsc{Choice}\ \frac{}{\neg(ps)\ \mid\ p(Cp)}\ \text{s term of suitable type}$$

Tableau rules for Choice

- G. Mints: "Cut-Elimination for Simple Type Theory with an Axiom of Choice"; Journal of Symbolic Logic 64 (2), 479-485. 1999. (the paper is from 1996)
- Does not use λ-calculus but "Takeuti's style" / relational style (see next slide)
- Uses a proof system based on a sequent calculus
- Author proves completeness with Choice but without Cut
- Assume C p, C p' occur as subterms on the branch

$$\textsc{Choice}\ \frac{}{\neg(ps) \ \mid\ p(Cp)}\ \text{s term of suitable type}$$

$$\textsc{Choice}_{Ext}\ \frac{}{pa, \neg(p'a)\ \mid\ \neg(pa), p'a\ \mid\ Cp = Cp'}\ a \text{ fresh}$$

Tableau rules for Choice

- G. Mints: "Cut-Elimination for Simple Type Theory with an Axiom of Choice"; Journal of Symbolic Logic 64 (2), 479-485. 1999. (the paper is from 1996)
- Does not use λ-calculus but "Takeuti's style" / relational style (see next slide)
- Uses a proof system based on a sequent calculus
- Author proves completeness with Choice but without Cut
- Assume C p, C p' occur as subterms on the branch

$$\textsc{Choice}\ \frac{}{\neg(ps) \ \mid\ p(Cp)}\ \text{s term of suitable type}$$

$$\textsc{Choice}_{Ext}\ \frac{}{pa, \neg(p'a)\ \mid\ \neg(pa), p'a\ \mid\ Cp = Cp'}\ a \text{ fresh}$$

$$\text{MAT'}\ \frac{\alpha s_1 \ldots s_n, \neg\alpha t_1 \ldots t_n}{s_1 \neq t1\ \mid\ \ldots\ \mid\ s_n \neq t_n}\ \alpha \text{ variable or some C p}$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$
- Recall:

$$\neg C(\lambda x.x)$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$
- Recall:

$$\neg C(\lambda x.x)$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$
- Recall: $\textsc{Choice}\ \dfrac{}{\neg(ps) \ \mid\ p(Cp)}$ s term of suitable type

$$\neg C(\lambda x.x)$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$
- Recall: $\textsc{Choice}\ \dfrac{}{\neg(ps) \ \mid \ p(Cp)}$ s term of suitable type

$$\neg C(\lambda x.x)$$

$$\begin{array}{c|c} & (\lambda x.x)(C(\lambda x.x)) \end{array}$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$
- Recall: $\textsc{Choice}\ \dfrac{}{\neg(ps) \ \mid \ p(Cp)}$ s term of suitable type

$$\dfrac{\neg C(\lambda x.x)}{\begin{array}{c|c} \quad\quad & (\lambda x.x)(C(\lambda x.x)) \\ & C(\lambda x.x) \end{array}}$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$
- Recall: $\textsc{Choice}\ \dfrac{}{\neg(ps)\ \mid\ p(Cp)}$ s term of suitable type

$$\begin{array}{c|c} \multicolumn{2}{c}{\neg C(\lambda x.x)} \\ \hline \neg((\lambda x.x)\top) & (\lambda x.x)(C(\lambda x.x)) \\ & C(\lambda x.x) \end{array}$$

Example for a tableau proof with Choice

- We want to prove the validity of $C_o(\lambda x.x)$

- Recall: $\textsc{Choice}\ \dfrac{}{\neg(ps) \mid p(Cp)}$ s term of suitable type

$$\dfrac{\neg C(\lambda x.x)}{\begin{array}{c|c} \neg((\lambda x.x)\top) & (\lambda x.x)(C(\lambda x.x)) \\ \neg\top & C(\lambda x.x) \end{array}}$$

Current state

- I'm "translating" the cut-freeness proof by Mints to the lambda calculus / tableaux
- Why is that not so easy?

Current state

- I'm "translating" the cut-freeness proof by Mints to the lambda calculus / tableaux

- Why is that not so easy?

$$(\epsilon)\ \frac{\Gamma \to \triangle, A(V) \qquad A(\epsilon x A(x)), \Gamma \to \triangle}{\Gamma \to \triangle}$$

$$(ext\,\epsilon)\ \frac{A(a), \Gamma \to \triangle, B(a) \quad B(a), \Gamma \to \triangle, A(a) \quad \forall z(\epsilon x A(x)[z] \leftrightarrow \epsilon y B(y)[z]), \Gamma \to \triangle}{\Gamma \to \triangle}$$

$$(ext)\ \frac{V_1(a), \Gamma \to \triangle, V_2(a) \qquad V_2(a), \Gamma \to \triangle, V_1(a)}{\alpha[V_1], \Gamma \to \triangle, \alpha[V_2]}$$

Thank you!

References

- Chad E. Brown, G. Smolka: "Terminating Tableaux for the Basic Fragment of Simple Type Theory". TABLEAUX 2009:138-151, Springer LNCS 5607.
- Chad E. Brown, G. Smolka: "Extended First-Order Logic". TPHOLs 2009, Springer LNCS 5674.
- Chad E. Brown, G. Smolka: "Complete Cut-Free Tableaux for Equational Simple Type Theory". 2009.
- G. Mints: Cut-Elimination for Simple Type Theory with an Axiom of Choice. The Journal of Symbolic Logic, Vol. 64, No. 2., pp. 479-485. 1999.
- G. Takeuti: "Proof Theory (Studies in Logic and the Foundations of Mathematics)". Elsevier Science Ltd, 2 Rev Sub edition. 1987.