Mechanised constructive reverse mathematics: soundness and completeness of bi-intuitionistic logic

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A typical article in logic consists in providing a proof system, a semantics, and establishing that both capture the same logic via soundness and completeness results. Oftentimes, the proofs of these results are lengthy, bureaucratic, technical, and difficult. As a consequence, these arguments can remain partial, as many details need to be elided, and be prone to errors. Additionally, the overall opacity of these proofs leads to a poor understanding of their constitution. In particular, the non-constructive principles they rely on are rarely detected or known.

The history of bi-intuitionistic logic (BIL) showcases the points we just made. This logic, which extends intuitionistic logic with a binary operator $\dot{-}\!\!<$ dual to $\dot{\to}$, was first studied in depth by Rauszer in the 1970s [9, 10, 11, 12, 13, 14, 15, 16]. In her work, Rauszer presented what she claimed to be a proof of completeness connecting a Kripke semantics to an axiomatic system (à la Hilbert) for BIL. However, almost 50 years later Goré and Shillito discovered critical errors in this proof, pertaining to the status of the deduction theorem [3]. In their paper, they provided a correct pen-and-paper completeness proof for the propositional fragment of BIL which was mechanised in Coq only later by Shillito [17].

Without a doubt, the mechanisation of Shillito brings a qualitative change to the pen-and-paper completeness (and soundness) proof: it makes it unquestionable, as each single step in the proof is now unambiguously mechanically checked. In addition to that, the mechanisation helps approximate the level of non-constructivity involved from above: the library `Classical` for classical logic is invoked in various places, supporting the multiple uses of the law of excluded-middle (LEM) throughout the proof.

As a consequence, we can safely assert that the result of completeness for BIL has a *de facto* non-constructive proof. The perspective of constructive reverse mathematics [7, 1] then suggests the following approximation from below: which non-constructive principles, if any, are *necessary* for this completeness result? Incidentally, a similar question can already be asked about the soundness result, as some axioms of BIL concerning $\dot{-}\!\!<$ have a classical flavour that is not interpretable in a fully constructive meta-logic.

Consequently, this is the topic of our work: we investigate the minimal non-constructive principles used to show soundness and completeness of BIL. Our investigations are led using the interactive theorem prover Coq, which is a perfect framework for constructive reverse mathematics, as it is based on a rather agnostic constructive type theory that allows fine sub-classical distinctions. In particular, constructive proofs in Coq can be extracted to executable programs, which for the specific case of completeness take the form of reification algorithms turning meta-level (semantic) proof terms into object-level (syntactic) derivations. This method has already been used successfully in the case of first-order logic and related formalisms [5, 2, 4, 8, 6] and we show that similar observations apply to BIL.

More concretely, we consider an alternative formulation of the Kripke semantics for BIL, which is classically equivalent to the traditional version, but intuitionistically weaker. For this formulation, we show that arbitrary soundness and completeness are equivalent to LEM. Soundness can be given a constructive proof if we restrict our attention to classical models, i.e. models which have a classical forcing relation. On the other hand, as intermediate steps of the completeness proof, quasi-completeness is equivalent to WLEMS while model existence is equivalent to the stronger WLEM. Finally, completeness restricted to enumerable contexts is connected to MP.

Syntax and generalised Hilbert system First, we introduce the syntax of BIL in BNF notation by

$$\varphi ::= n \in \mathcal{V} \mid \dot{\bot} \mid \dot{\top} \mid \varphi \mathbin{\dot{\land}} \varphi \mid \varphi \mathbin{\dot{\lor}} \varphi \mid \varphi \mathbin{\dot{\to}} \varphi \mid \varphi \mathbin{\dot{\prec}} \varphi$$

with two defined connectives $\dot{\neg}\varphi := (\varphi \mathbin{\dot{\to}} \dot{\bot})$ and $\dot{\sim}\varphi := (\dot{\top} \mathbin{\dot{\prec}} \varphi)$. The added binary operator $\dot{\prec}$ is the dual of $\dot{\to}$ and is usually read as "φ excludes ψ". Also, the defined connective $\dot{\sim}$ is called "weak negation".

Secondly, the generalised Hilbert calculus [3] BIH for BIL extends the one for intuitionistic logic with the axioms A_{11} to A_{14} and the rule (wDN), shown below. We write $\Gamma \vdash \varphi$ if φ is provable in BIH from context Γ.

$$\begin{array}{llll} A_{11} & \varphi \mathbin{\dot{\to}} (\psi \mathbin{\dot{\lor}} (\varphi \mathbin{\dot{\prec}} \psi)) & A_{13} & ((\varphi \mathbin{\dot{\prec}} \psi) \mathbin{\dot{\prec}} \chi) \mathbin{\dot{\to}} (\varphi \mathbin{\dot{\prec}} (\psi \mathbin{\dot{\lor}} \chi)) \\ A_{12} & (\varphi \mathbin{\dot{\prec}} \psi) \mathbin{\dot{\to}} \dot{\sim}(\varphi \mathbin{\dot{\to}} \psi) & A_{14} & \dot{\neg}(\varphi \mathbin{\dot{\prec}} \psi) \mathbin{\dot{\to}} (\varphi \mathbin{\dot{\to}} \psi) \end{array} \qquad \frac{\emptyset \vdash \varphi}{\Gamma \vdash \dot{\neg}\dot{\sim}\varphi}\ \text{(wDN)}$$

Alternative Kripke semantics The traditional semantics for BIL [16], which was used in the initial formalisation [17], is defined using Kripke models. More precisely, the models for bi-intuitionistic logic are identical to the ones of intuitionistic logic, as shown below.

Definition 1. *A Kripke model $\mathcal{M}$ is a tuple $(W, \leq, I)$, where $(W, \leq)$ is a poset and $I : \mathcal{V} \to \mathcal{P}(W)$ is a* persistent *interpretation function:*

$$\forall v, w \in W. \forall p \in \mathcal{V}. (w \leq v \land w \in I(p)) \to v \in I(p)$$

In addition to that, the forcing relation for the traditional semantics for BIL is the one of intuitionistic logic extended to $\dot{\prec}$ as follows.

Definition 2. *Given a Kripke model $\mathcal{M} = (W, \leq, I)$, we extend the usual forcing relation $\mathcal{M}, w \Vdash \varphi$ to exclusion (first recalling the rule for implication for comparison):*

$$\begin{array}{lll} \mathcal{M}, w \Vdash \varphi \mathbin{\dot{\to}} \psi & \textit{if} & \forall v \geq w.\, \mathcal{M}, v \Vdash \varphi \to \mathcal{M}, v \Vdash \psi \\ \mathcal{M}, w \Vdash \varphi \mathbin{\dot{\prec}} \psi & \textit{if} & \exists v \leq w.\, \mathcal{M}, v \Vdash \varphi \land \mathcal{M}, v \nVdash \psi \end{array}$$

While the alternative Kripke semantics we define uses the same models, it modifies the forcing relation in the clause for $\dot{\prec}$ in the following (constructively weaker but classically equivalent) way.

$$\mathcal{M}, w \Vdash \varphi \mathbin{\dot{\prec}} \psi \quad \text{if} \quad \neg\forall v \leq w.\, M, v \Vdash \varphi \to \mathcal{M}, v \Vdash \psi$$

The notion of (local) semantic consequence relation is then derived as follows.

$$\Gamma \models \varphi \quad \text{if} \quad \forall \mathcal{M}. \forall w. (\mathcal{M}, w \Vdash \Gamma \to \mathcal{M}, w \Vdash \varphi)$$

Constructive analysis We define results linking the semantic consequence relations and BIL.

Soundness	$=_{def}$	$\Gamma \vdash \varphi \to \Gamma \models \varphi$
Model existence	$=_{def}$	$\neg(\Gamma \vdash \varphi) \to \exists \mathcal{M}. \exists w. \mathcal{M}, w \Vdash \Gamma$
Quasi-completeness	$=_{def}$	$\Gamma \models \varphi \to \neg\neg(\Gamma \vdash \varphi)$
Completeness	$=_{def}$	$\Gamma \models \varphi \to \Gamma \vdash \varphi$

Note that we can specify all these notions by restricting the type of models $\mathcal{M}$ (e.g. classical) and contexts Γ (e.g. arbitrary, enumerable, finite). In addition to constructively provable soundness for classical models, we have the following results, where Markov's principle (MP) can be stated by $\forall f : \mathbb{N} \to \mathbb{N}. \neg\neg(\exists n. fn = 0) \to \exists n. fn = 0$ while weak excluded middle shift (WLEMS) is equivalent to $\forall PQ. (\forall n. \neg\neg(\neg Pn \lor \neg Qn)) \to \neg\neg(\forall n. \neg Pn \lor \neg Qn)$ and EWLEMS is then the restriction to enumerable P and Q.

Soundness	$\Leftrightarrow$	LEM	$=_{def}$	$\forall P.\, P \vee \neg P$
Model Existence	$\Leftrightarrow$	WLEM	$=_{def}$	$\forall P.\, \neg P \vee \neg\neg P$
Quasi-Completeness	$\Leftrightarrow$	WLEMS	$=_{def}$	$\forall P. \neg\neg \forall n : \mathbb{N}. \neg Pn \vee \neg\neg Pn$
Arbitrary Completeness	$\Leftrightarrow$	LEM		
WLEMS+MP	$\Rightarrow$	Enumerable Completeness	$\Rightarrow$	EWLEMS+MP

A natural extension of our work consists in investigating the first-order case. However, there is currently no proof of completeness for first-order bi-intuitionistic logic. Notably, the proofs by Rauszer [16] are suffering from the same issues as in the propositional case, but could not yet be fixed for technical reasons [17]. So, to tackle the first-order case we need to first obtain any proof, regardless of its degree of non-constructivity, and then proceed as we did for the propositional case.

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