New Observations on the Constructive Content of First-Order Completeness Theorems

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Abstract

We report on some new observations regarding the constructive reverse mathematics of first-order completeness theorems. When conducted in a constructive type theory such as the calculus of inductive constructions (CIC), many different formulations of completeness can be distinguished and analysed regarding their sufficient and necessary non-constructive assumptions. As the main new result, we identify a principle we dub WLEMS at the intersection of double-negation shift and weak excluded middle, exactly capturing the non-constructivity needed for object-level disjunctions. The observations reported here are part of an ongoing general attempt at a systematic classification of the independent ingredients contributing to the non-constructivity of completeness theorems.

Background Completeness of a logic states that every formula φ semantically entailed by a (possibly infinite) context Γ, denoted by $\Gamma \vDash \varphi$, also admits a syntactical derivation via the rules of a suitable deduction calculus, denoted by $\Gamma \vdash \varphi$. After the discovery that Gödel's completeness theorem of first-order logic [5] relies on Markov's principle (MP) [11], this first-order completeness has been of ongoing interest for the programmes of reverse mathematics [4, 14] and constructive reverse mathematics [9].

Seeking to pin down the exact (non-constructive) assumptions required by the analysed theorems, these programmes are carried out in purposefully weak (constructive) logical systems that allow fine distinctions of logical strength. Among the known results regarding completeness are, besides the mentioned connection to MP, equivalences to the weak König's lemma [15], the weak fan theorem [12], as well as the Boolean prime ideal theorem [6].

Formulations of Completeness Working in CIC [2, 13] and modelling first-order logic in an established way [10], we distinguish the following forms of completeness:

- Completeness: $\forall \varphi, \Gamma.\, \Gamma \vDash \varphi \to \Gamma \vdash \varphi$
- Quasi-Completeness: $\forall \varphi, \Gamma.\, \Gamma \vDash \varphi \to \neg\neg(\Gamma \vdash \varphi)$
- Model Existence: $\forall \varphi, \Gamma.\, \Gamma \nvdash \varphi \to \exists M.\, M \vDash \Gamma \land M \nvDash \varphi$

Usual (Henkin-style) proofs establish model existence first, from which then only quasi-completeness follows constructively, given its additional double negation. Next to these forms of completeness, there are four other dimension contributing to the (non)-constructivity, namely

- The complexity of the context (e.g., finite, decidable, enumerable, arbitrary),
- The cardinality of the signature (e.g., countable, uncountable),
- The syntax fragment (e.g., propositional, minimal, negative, full), and
- The representation of the semantics (e.g., Boolean, decidable, propositional).

In this abstract we discuss the cases of quasi-completeness and model existence for arbitrary contexts over a countable signature, regarding the full syntax (including the critical case of disjunctions) and propositional semantics, which was left open in previous work [8, 3, 7].

Main Observations For the mentioned target formulation of completeness we identify the following principle we call *weak excluded middle shift* (WLEMS):[1]

$$\forall p : \mathbb{N} \to \mathbb{P}.\, \neg\neg(\forall n.\, \neg p\, n \lor \neg\neg p\, n)$$

Note that WLEMS follows both from double-negation shift (DNS),[2] as this would allow to push the outer double negation through the universal quantifier, and from weak excluded middle (WLEM),[3] as this allows the inner classical case distinctions. Both DNS and WLEM give rise to proofs of quasi-completeness themselves, however subsumed by the following main observation:

Theorem 1. *Quasi-completeness (for the said setting) is equivalent to WLEMS.*

Sketch. First assume $\Gamma \vDash \varphi$ and $\Gamma \nvdash \varphi$ for a contradiction. The latter allows to constructively extend Γ to Δ with several closure properties (neglecting the usual treatment of quantifiers):

- Relative Consistency: $\Delta \nvdash \varphi$
- Deductive Closure: $\forall \psi.\, \Delta \vdash \psi \to \psi \in \Delta$
- Stability: $\forall \psi.\, \neg\neg(\psi \in \Delta) \to \psi \in \Delta$
- Quasi-Primeness: $\forall \psi, \psi'.\, \psi \dot\lor \psi' \in \Delta \to \neg\neg(\psi \in \Delta \lor \psi' \in \Delta)$

Combining WLEMS and stability, we can pull the double negation in quasi-primeness to the front and, given the goal to be deriving a contradiction, obtain actual primeness, which is quasi-primeness without any double negations. In a usual completeness proof, primeness is exactly the property needed to verify that the syntactic model M_Δ arising from Δ satisfies both $M_\Delta \vDash \Gamma$ and $M_\Delta \nvDash \varphi$, in contradiction to the assumption $\Gamma \vDash \varphi$.

Conversely given $p : \mathbb{N} \to \mathbb{P}$ with $\neg(\forall n.\, \neg p\, n \lor \neg\neg p\, n)$ for a contradiction, we consider

$$\Gamma := \{P_n \dot\lor \dot\neg P_n \mid n : \mathbb{N}\} \cup \{P_n \mid p\, n\} \cup \{\dot\neg P_n \mid \neg p\, n\}$$

using countably many propositional variables P_n. Applying quasi completeness for $\varphi := \dot\bot$, we are left to show that $\Gamma \vDash \dot\bot$ and that Γ is consistent. The latter is possible using a suitable model and soundness, the former boils down to assuming a model M with $M \vDash \Gamma$ and then showing $\forall n.\, \neg p\, n \lor \neg\neg p\, n$ by inspecting the choices $M \vDash P_n \dot\lor \dot\neg P_n$ made by the model. □

Complementing the previous equivalence, we also observe:

Theorem 2. *Model existence (for the said setting) is equivalent to WLEM.*

Sketch. WLEM is enough to show that stable quasi-prime theories are actually prime, yielding model existence as above. Conversely given $p : \mathbb{P}$, model existence for the consistent context $\Gamma := \{P_0 \dot\lor \dot\neg P_0\} \cup \{P_0 \mid p\} \cup \{\dot\neg P_0 \mid \neg p\}$ yields the desired case distinction $\neg p \lor \neg\neg p$. □

Outlook First, although presented here for classical first-order logic, we expect that the same results hold for intuitionistic first-order logic. Actually, we suspect that the characterisation of disjunction using WLEMS and WLEM is universal enough to apply also to other logics like modal logic or bi-intuitionistic logic. Secondly, it would be desirable to develop an abstract framework for completeness proofs, orthogonalising the different dimensions of non-constructivity and generalising over the concrete specifics of the analysed logic. Thirdly, we want to investigate the exact correlation of WLEMS and formulations of the weak fan theorem, especially regarding Berger's decomposition of the latter [1].

[1] Equivalent to *disjunctive double-negation shift*: $\forall pq : \mathbb{N} \to \mathbb{P}.\, (\forall n.\, \neg\neg(\neg p\, n \lor \neg q\, n)) \to \neg\neg(\forall n.\, \neg p\, n \lor \neg q\, n)$ and mentioned in more general form as M° quantifying over arbitrary domains by Umezawa [16].

[2] $\forall X.\, \forall p : X \to \mathbb{P}.\, (\forall x.\, \neg\neg p\, x) \to \neg\neg(\forall x.\, p\, x)$

[3] $\forall p : \mathbb{P}.\, \neg p \lor \neg\neg p$

References

[1] Josef Berger. A decomposition of Brouwer's fan theorem. *Journal of Logic and Analysis*, 1, 2009.

[2] Thierry Coquand. *The calculus of constructions*. PhD thesis, INRIA, 1986.

[3] Yannick Forster, Dominik Kirst, and Dominik Wehr. Completeness theorems for first-order logic analysed in constructive type theory. In *International Symposium on Logical Foundations of Computer Science*. Springer, 2020.

[4] Harvey Friedman. Some systems of second order arithmetic and their use. In *Proceedings of the international congress of mathematicians (Vancouver, BC, 1974)*, volume 1, pages 235–242. Citeseer, 1975.

[5] Kurt Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatshefte für Mathematik und Physik*, 37:349–360, 1930. URL: https://zbmath.org/?q=an%3A56.0046.04.

[6] Leon Henkin. Metamathematical theorems equivalent to the prime ideal theorem for boolean algebras. *Bulletin AMS*, 60:387–388, 1954.

[7] Hugo Herbelin. Computing with Gödel's completeness theorem: Weak fan theorem, Markov's principle and double negation shift in action. http://pauillac.inria.fr/~herbelin/talks/chocola22.pdf, 2022.

[8] Hugo Herbelin and Danko Ilik. An analysis of the constructive content of Henkin's proof of Gödel's completeness theorem. 2016.

[9] Hajime Ishihara. Reverse Mathematics in Bishop's Constructive Mathematics. *Philosophia Scientae*, pages 43–59, 2006. doi:10.4000/philosophiascientiae.406.

[10] Dominik Kirst, Johannes Hostert, Andrej Dudenhefner, Yannick Forster, Marc Hermes, Mark Koch, Dominique Larchey-Wendling, Niklas Mück, Benjamin Peters, Gert Smolka, and Dominik Wehr. A Coq library for mechanised first-order logic. In *Coq Workshop*, 2022.

[11] Georg Kreisel. On weak completeness of intuitionistic predicate logic. *The Journal of Symbolic Logic*, 27(2):139–158, 1962.

[12] Victor N Krivtsov. Semantical completeness of first-order predicate logic and the weak fan theorem. *Studia Logica*, 103(3):623–638, 2015.

[13] Christine Paulin-Mohring. Inductive definitions in the system Coq - rules and properties. In *International Conference on Typed Lambda Calculi and Applications*, pages 328–345. Springer, 1993.

[14] Stephen G. Simpson. Reverse mathematics. In *Proc. Symposia Pure Math*, volume 42, pages 461–471, 1985.

[15] Stephen G. Simpson. *Subsystems of second order arithmetic*, volume 1. Cambridge University Press, 2009.

[16] Toshio Umezawa. On logics intermediate between intuitionistic and classical predicate logic. *The Journal of Symbolic Logic*, 24(2):141–153, 1959.