New Observations on the Constructive Content of First-Order Completeness Theorems

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Abstract

We report on some new observations regarding the constructive reverse mathematics of first-order completeness theorems. When conducted in a constructive type theory such as the calculus of inductive constructions (CIC), many different formulations of completeness can be distinguished and analysed regarding their sufficient and necessary non-constructive assumptions. As the main new result, we identify a principle we dub WLEMS at the intersection of double-negation shift and weak excluded middle, exactly capturing the non-constructivity needed for object-level disjunctions. The observations reported here are part of an ongoing general attempt at a systematic classification of the independent ingredients contributing to the non-constructivity of completeness theorems.

Background

Completeness of a logic states that every formula $\varphi$ semantically entailed by a (possibly infinite) context $\Gamma$, denoted by $\Gamma \models \varphi$, also admits a syntactical derivation via the rules of a suitable deduction calculus, denoted by $\Gamma \vdash \varphi$. After the discovery that Gödel’s completeness theorem of first-order logic [5] relies on Markov’s principle (MP) [11], this first-order completeness has been of ongoing interest for the programmes of reverse mathematics [4, 14] and constructive reverse mathematics [9].

Seeking to pin down the exact (non-constructive) assumptions required by the analysed theorems, these programmes are carried out in purposefully weak (constructive) logical systems that allow fine distinctions of logical strength. Among the known results regarding completeness are, besides the mentioned connection to MP, equivalences to the weak König’s lemma [15], the weak fan theorem [12], as well as the Boolean prime ideal theorem [6].

Formulations of Completeness

Working in CIC [2, 13] and modelling first-order logic in an established way [10], we distinguish the following forms of completeness:

- Completeness: $\forall \varphi, \Gamma \models \varphi \rightarrow \Gamma \vdash \varphi$
- Quasi-Completeness: $\forall \varphi, \Gamma \models \varphi \rightarrow \neg \neg (\Gamma \vdash \varphi)$
- Model Existence: $\forall \varphi, \Gamma \not\vdash \varphi \rightarrow \exists M. M \models \Gamma \land M \not\models \varphi$

Usual (Henkin-style) proofs establish model existence first, from which then only quasi-completeness follows constructively, given its additional double negation. Next to these forms of completeness, there are four other dimension contributing to the (non)-constructivity, namely

- The complexity of the context (e.g., finite, decidable, enumerable, arbitrary),
- The cardinality of the signature (e.g., countable, uncountable),
- The syntax fragment (e.g., propositional, minimal, negative, full), and
- The representation of the semantics (e.g., Boolean, decidable, propositional).

In this abstract we discuss the cases of quasi-completeness and model existence for arbitrary contexts over a countable signature, regarding the full syntax (including the critical case of disjunctions) and propositional semantics, which was left open in previous work [8, 3, 7].
Main Observations For the mentioned target formulation of completeness we identify the following principle we call weak excluded middle shift (WLEMS):

\[ \forall p : \mathbb{N} \rightarrow \mathbb{P}. \neg(\forall n. \neg p \lor \neg \neg p) \]

Note that WLEMS follows both from double-negation shift (DNS),\(^2\) as this would allow to push the outer double negation through the universal quantifier, and from weak excluded middle (WLEM),\(^3\) as this allows the inner classical case distinctions. Both DNS and WLEM give rise to proofs of quasi-completeness themselves, however subsumed by the following main observation:

**Theorem 1.** Quasi-completeness (for the said setting) is equivalent to WLEMS.

*Sketch.* First assume \( \Gamma \vdash \varphi \) and \( \Gamma \nvdash \varphi \) for a contradiction. The latter allows to constructively extend \( \Gamma \) to \( \Delta \) with several closure properties (neglecting the usual treatment of quantifiers):

- Relative Consistency: \( \Delta \nvdash \varphi \)
- Deductive Closure: \( \forall \psi. \Delta \vdash \psi \rightarrow \psi \in \Delta \)
- Stability: \( \forall \psi. \neg(\psi \in \Delta) \rightarrow \psi \in \Delta \)
- Quasi-Primeness: \( \forall \psi, \psi'. \psi \lor \psi' \in \Delta \rightarrow \neg(\psi \in \Delta \lor \psi' \in \Delta) \)

Combining WLEMS and stability, we can pull the double negation in quasi-primeresness to the front and, given the goal to be deriving a contradiction, obtain actual primeness, which is quasi-primeresness without any double negations. In a usual completeness proof, primeness is exactly the property needed to verify that the syntactic model \( M_{\Delta} \) arising from \( \Delta \) satisfies both \( M_{\Delta} \vdash \Gamma \) and \( M_{\Delta} \nvdash \varphi \), in contradiction to the assumption \( \Gamma \vdash \varphi \).

Conversely given \( p : \mathbb{N} \rightarrow \mathbb{P} \) with \( \neg(\forall n. \neg p \lor \neg \neg p) \) for a contradiction, we consider

\[ \Gamma := \{ P_n \lor P_n \mid n : \mathbb{N} \} \cup \{ P_n \mid p n \} \cup \{ \neg P_n \mid \neg p n \} \]

using countably many propositional variables \( P_n \). Applying quasi completeness for \( \varphi := \bot \), we are left to show that \( \Gamma \vdash \bot \) and that \( \Gamma \) is consistent. The latter is possible using a suitable model and soundness, the former boils down to assuming a model \( M \) with \( M \vdash \Gamma \) and then showing \( \forall n. \neg p n \lor \neg \neg p n \) by inspecting the choices \( M \vdash P_n \lor P_n \) made by the model. \( \square \)

Complementing the previous equivalence, we also observe:

**Theorem 2.** Model existence (for the said setting) is equivalent to WLEM.

*Sketch.* WLEM is enough to show that stable quasi-prime theories are actually prime, yielding model existence as above. Conversely given \( p : \mathbb{P} \), model existence for the consistent context \( \Gamma := \{ P_b \lor P_b \} \cup \{ P_b \mid p \} \cup \{ \neg P_b \mid \neg p \} \) yields the desired case distinction \( \neg p \lor \neg \neg p \). \( \square \)

Outlook First, although presented here for classical first-order logic, we expect that the same results hold for intuitional first-order logic. Actually, we suspect that the characterisation of disjunction using WLEMS and WLEM is universal enough to apply also to other logics like modal logic or bi-intuitionistic logic. Secondly, it would be desirable to develop an abstract framework for completeness proofs, orthogonalising the different dimensions of non-constructivity and generalising over the concrete specifics of the analysed logic. Thirdly, we want to investigate the exact correlation of WLEMS and formulations of the weak fan theorem, especially regarding Berger’s decomposition of the latter \([1]\).

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\(^{1}\)Equivalent to disjunctive double-negation shift: \( \forall p q : \mathbb{N} \rightarrow \mathbb{P}. (\forall n. \neg p n \lor \neg q n) \rightarrow \neg(\forall n. \neg p n \lor \neg q n) \)

\(^{2}\)\( \forall X : \mathbb{P} : X \rightarrow \mathbb{P}. (\forall x. \neg \lnot p x) \rightarrow \neg(\forall x. p x) \)

\(^{3}\)\( \forall p : \mathbb{P}. \neg p \lor \neg \lnot p \)
References


