Markov's Principles in Constructive Type Theory

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Abstract

Markov's principle (MP) is an axiom in some varieties of constructive mathematics, stating that Σ^0_1 propositions (i.e. existential quantification over a decidable predicate on $\mathbb{N}$) are stable under double negation. However, there are various non-equivalent definitions of decidable predicates and thus Σ^0_1 in constructive foundations, leading to non-equivalent Markov's principles. This fact is often overlooked and leads to confusion: At the time of writing, both Wikipedia and nlab claim propositions to be equivalent to MP, which are however only respectively equivalent to two non-equivalent forms of MP.

We give three variants of MP in constructive type theory, along with respective equivalence proofs to different formulations of Post's theorem ("Σ^0_1-predicates with complement in Σ^0_1 are decidable"), stability of termination of computations, the statement that an extended natural number is finite if it is not infinite, and to completeness of natural deduction w.r.t. Tarski semantics over the $(\forall, \to, \bot)$-fragment of classical first-order logic for Σ^0_1-theories. The first definition ($\mathsf{MP}_\mathbb{P}$) uses a purely logical definition of Σ^0_1 for predicates $\mathbb{N} \to \mathbb{P}$, while the second one ($\mathsf{MP}_\mathbb{B}$) relies on type-theoretic functions $\mathbb{N} \to \mathbb{B}$, and the third one ($\mathsf{MP}_\mathsf{PR}$) on a model of computation.

We conclude with the – to the best of our knowledge – first proof that $\mathsf{MP}_\mathbb{B}$ is *not* equivalent to MP_PR using a model via Cohen and Rahli's $\mathrm{TT}^\square_\mathcal{C}$, and pose the open question how to separate $\mathsf{MP}_\mathbb{P}$ from $\mathsf{MP}_\mathbb{B}$ – where the model would have to invalidate unique choice.

Definitions We work in constructive type theory with a universe of propositions $\mathbb{P}$, e.g. in the calculus of inductive constructions (CIC). We define three variants of Markov's principle:

$$\mathsf{MP}_\mathbb{P} := \forall A : \mathbb{N} \to \mathbb{P}.\ (\forall n.\ An \lor \neg An) \to \neg\neg(\exists n.\ An) \to (\exists n.\ An)$$

$$\mathsf{MP}_\mathbb{B} := \forall f : \mathbb{N} \to \mathbb{B}.\ \neg\neg(\exists n.\ fn = \mathsf{true}) \to (\exists n.\ fn = \mathsf{true})$$

$$\mathsf{MP}_\mathsf{PR} := \forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{primitive\text{-}recursive}\ f \to \neg\neg(\exists n.\ fn = \mathsf{true}) \to (\exists n.\ fn = \mathsf{true})$$

We write MP_PR following Troelstra and van Dalen [12]. Due to the Kleene normal form theorem [7], any principle replacing primitive recursiveness with computability in any Turing complete model is equivalent, e.g. called MP_L in [4] after the weak call-by-value λ-calculus L [6].

Note that $\mathsf{MP}_\mathbb{P}$ implies $\mathsf{MP}_\mathbb{B}$, which in turn implies MP_PR. The first implication is an equivalence given the axiom of (type-theoretic) unique choice, i.e. if $\forall R : \mathbb{N} \to \mathbb{B} \to \mathbb{P}.$ $(\forall n : \mathbb{N}. \exists! b : \mathbb{B}. Rnb) \to \exists f : \mathbb{N} \to \mathbb{B}. \forall n.\ Rn(fn)$ holds, because then any such $A : \mathbb{N} \to \mathbb{P}$ gives rise to a decider of type $\mathbb{N} \to \mathbb{B}$. The second implication is an equivalence under CT ("Church's thesis" [9]), i.e. if the proposition $\forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{computable}\ f$ holds. $\mathsf{MP}_\mathbb{P}$ is consistent because it is a consequence of the law of excluded middle (LEM). $\mathsf{MP}_\mathbb{B}$ is proved independent from type theory by Mannaa and Coquand [2] as well as Pedrót and Tabareau [10], and MP_L by Forster, Kirst, and Wehr [5]. In a (weak) type theory such as CIC, both the unique choice axiom from above and CT are independent. In many constructive foundations (CZF, IZF, HoTT, or in MLTT with $\exists$ as Σ), unique choice is a theorem, but CT remains independent. Since in all these foundations $\exists n : \mathbb{N}.\ fn = \mathsf{true}$ implies $\Sigma n : \mathbb{N}.\ fn = \mathsf{true}$, stating MP with Σ or $\exists$ is equivalent.

On Σ^0_1 Predicates A predicate $p : X \to \mathbb{P}$ is stable under double negation if $\forall x.\ \neg\neg px \to px$, and is Σ^0_1 if there exists a decidable predicate $A : X \to \mathbb{N} \to \mathbb{P}$ such that $\forall x.\ px \leftrightarrow \exists n.\ Axn$. Now if decidable predicates $A : X \to \mathbb{N} \to \mathbb{P}$ only need to fulfill $\forall xn.\ Axn \lor \neg Axn$, then stability of Σ^0_1 predicates is equivalent to $\mathsf{MP}_\mathbb{P}$. If however decidable predicates are associated with a function of type $X \to \mathbb{N} \to \mathbb{B}$, stability of Σ^0_1 predicates is equivalent to $\mathsf{MP}_\mathbb{B}$. And if decidable predicates are associated with a *computable* function of that type, it is equivalent to MP_PR.

Post's Theorem (PT) [11] states that Σ^0_1 predicates with complement in Σ^0_1 are decidable. With decidable predicates defined using type-theoretic functions, PT is equivalent to $\mathsf{MP}_{\mathbb{B}}$ [12], formalised in Coq by Forster, Kirst, and Smolka [3]. With decidable predicates defined using computable functions, PT is equivalent to $\mathsf{MP}_{\mathsf{PR}}$, formalised in Coq by Forster and Smolka [6]. With the logical definition, PT is equivalent to $\mathsf{MP}_{\mathbb{P}}$, a proof we contribute with this abstract.

Termination of Computation It is folklore that "a computation halts if it does not run forever" is equivalent to MP. Taking a computation as a Σ^0_1 relation $\mathbb{N}{\to}\mathbb{N}{\to}\mathbb{P}$, the three respective definitions of Σ^0_1 indeed render this equivalent to the respective version of MP. In particular, the statement "a Turing machine halts if it does not run forever" is equivalent to $\mathsf{MP}_{\mathsf{PR}}$.

Extended Natural Numbers One can model the extension of $\mathbb{N}$ with a point of infinity as monotonous infinite sequences of truth values b_i (if b_i then b_j holds for $j \geq i$). MP is equivalent to "an extended natural number which is not infinite is finite", precisely to $\mathsf{MP}_{\mathbb{P}}$ if sequences are defined as predicates $\mathbb{N} \to \mathbb{P}$, and to $\mathsf{MP}_{\mathbb{B}}$ if defined as functions $\mathbb{N} \to \mathbb{B}$. Defining sequences as *computable* functions $\mathbb{N} \to \mathbb{B}$ is unusual, but would be equivalent to $\mathsf{MP}_{\mathsf{PR}}$.

First-order Completeness It was already known to Gödel that completeness of natural deduction w.r.t. Tarski-semantics over the $(\forall, \to, \bot)$-fragment of classical first-order logic is equivalent to $\mathsf{MP}_{\mathsf{PR}}$ [8]. The result can be extended to Σ^0_1-theories, but again the definition of Σ^0_1 is crucial. The equivalences to $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{MP}_{\mathsf{PR}}$ are proved in Coq by Forster, Kirst, and Wehr [4], we contribute the respective (Coq) proof for $\mathsf{MP}_{\mathbb{P}}$.

$\mathbf{TT}^{\square}_{\mathcal{C}}$ is a general framework for type theories modeled through an abstract modality $\square$ and parameterised by a type of time-progressing choice operators $\mathcal{C}$ due to Cohen and Rahli [1], which is formalised in Agda. Time-progression here means that $\mathrm{TT}^{\square}_{\mathcal{C}}$'s computation system includes stateful computations that can evolve non-deterministically over time (captured by a poset $\mathcal{W}$ of worlds), and that can change the state of the world. Instantiating $\square$ and $\mathcal{C}$ can either validate or invalidate axioms such as MP.

Separation of $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{MP}_{\mathsf{PR}}$ We prove that instantiating $\mathcal{C}$ with choice sequences and $\square$ with a Beth modality as in [1] yields a model validating constructively $\neg\mathsf{MP}_{\mathbb{B}}$, and, assuming LEM in the meta-theory, $\mathsf{MP}_{\mathsf{PR}}$. To do so, we translate the types $\mathbb{N}$ and $\mathbb{B}$ to the types Nat and Bool of possibly effectful terms with two properties: (1) if they compute to a value in a world, they compute to the same value in all extensions of that world; and (2) whenever they compute to a value, they leave the world unchanged. Such effectful terms do not satisfy $\mathsf{MP}_{\mathbb{B}}$ because f can be undetermined for all inputs and thus satisfy $\neg\neg(\exists n.\ fn = \mathsf{true})$ but not $\exists n.\ fn = \mathsf{true}$. However, primitive recursive functions can be encoded as natural numbers, and thus behave like a pure, effect-free function. Concretely, we have that

$$\forall (w : \mathcal{W}). w \not\models \mathbf{\Pi} f{:}\mathsf{Nat} \to \mathsf{Bool}.(\neg\neg{\downarrow}\mathbf{\Sigma} n{:}\mathsf{Nat}.f\ n = \mathsf{true}) \to {\downarrow}\mathbf{\Sigma} n{:}\mathsf{Nat}.f\ n = \mathsf{true}$$

Here, the $\downarrow$ operation discards the computational content of the dependent pair type $\mathbf{\Sigma}$. Furthermore, with LEM in the meta-theory, MP for *pure* (i.e. effect-free) functions is valid in *all* models in [1] (see mpp.lagda), with $\mathbf{\Pi}_{\boldsymbol{p}}$ letting f range over pure, effect-free terms only:

$$\forall (w : \mathcal{W}). w \models \mathbf{\Pi}_{\boldsymbol{p}} f{:}\mathsf{Nat} \to \mathsf{Bool}.(\neg\neg{\downarrow}\mathbf{\Sigma} n{:}\mathsf{Nat}.f\ n = \mathsf{true}) \to {\downarrow}\mathbf{\Sigma} n{:}\mathsf{Nat}.f\ n = \mathsf{true}$$

To show that this implies $\mathsf{MP}_{\mathsf{PR}}$, note that $\mathsf{MP}_{\mathsf{PR}}$ can be equivalently stated as

$$\forall (w : \mathcal{W}). w \models \mathbf{\Pi} m{:}\mathsf{Nat}.(\neg\neg{\downarrow}\mathbf{\Sigma} n{:}\mathsf{Nat}.\mathsf{eval}\ m\ n = \mathsf{true}) \to {\downarrow}\mathbf{\Sigma} n{:}\mathsf{Nat}.\mathsf{eval}\ m\ n = \mathsf{true}$$

where $\mathsf{eval} : \mathbb{N} \to \mathbb{N} \to \mathbb{B}$ is a pure function which interprets its first argument as the Gödelisation of a primitive recursive function f, and for any primitive recursive f there is m with $\forall n. fn = \mathsf{eval}\ m\ n$. Now whenever an effectful m evaluates to c in a world w, we have that $\mathsf{eval}\ c$ is a pure function for all $w' \sqsupseteq w$, making the pure form of MP applicable (see pure2.lagda).

Acknowledgements We have posed the question how to prove that $\mathsf{MP}_{\mathbb{B}}$ and $\mathsf{MP}_{\mathsf{PR}}$ are not equivalent to many people over the years, and want to thank Liron Cohen, Thierry Coquand, Hugo Herbelin, Hajime Ishihara, Pierre-Marie Pédrot, and Gert Smolka for discussions and useful hints about directions to chase. Furthermore, Martín Escardó provided useful feedback on a first draft of this abstract and the idea to extend the question to $\mathsf{MP}_{\mathbb{P}}$ came up in a discussion with Dirk Pattinson. Thank you!

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