# Intersection Type Systems Corresponding to Nominal Automata

## UdS Qualifying Exam / MFoCS Dissertation

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November 23, 2016

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- $\vdash K((\lambda x.xbcx)a)\Omega : q_I$ for a more complex program
- Theorem: $\forall \Gamma, s, q.\ \Gamma \vdash s : q \iff \exists n.\, s \Downarrow n \wedge n \in \mathcal{L}(\mathcal{A}, q)$

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...for nominal automata[2]:

- Well-behaved automata over infinite alphabets
- Equivariant properties independent of concrete names

$\Longrightarrow$ Contribute new instances for new automaton models

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# Lambda Calculus[3]

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**Theorem**

The (untyped) lambda calculus is Turing complete.

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However, some normal forms like $\lambda x.xx$ are untybable...

# Intersection Types[5]

Introduce (finite) type intersections $A \wedge B$ with rules:

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However, general type checking and typability become undecidable...

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Let $\mathbb{A}$ be a countable set of atomic names.
Then consider sets $X$ with actions $\cdot : \mathrm{Perm}(\mathbb{A}) \times X \to X$ such that

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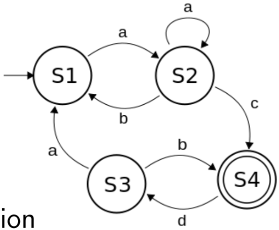
Examples: $\mathbb{A}$ itself, (finite) syntax over $\mathbb{A}$, singleton sets etc.

# Finite Automata

A finite automaton $\mathcal{A}$ consists of:

- $\Sigma$, a finite alphabet
- $Q$, a finite set of states
- $I \subseteq Q$, a finite subset of initial states
- $F \subseteq Q$, a finite subset of final states
- $\delta \subseteq Q \times A \times Q$, a finite transition relation



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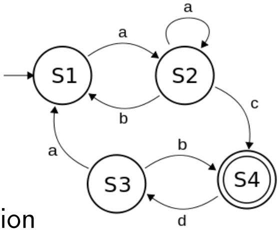


We write:

- $q \xrightarrow{a} q'$ for $a \in \Sigma$, $q, q' \in Q$ and $(q, a, q') \in \delta$
- $q \xrightarrow{w} q'$ for $w \in \Sigma^*$ and the reflexive-transitive closure of $\delta$
- $w \in \mathcal{L}(\mathcal{A})$ for $q \xrightarrow{w} q'$ with $q \in I$ and $q' \in F$

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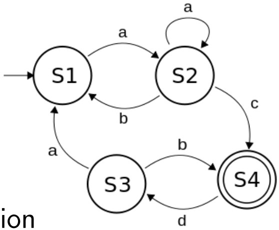


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- $w \in \mathcal{L}(\mathcal{A})$ for $q \xrightarrow{w} q'$ with $q \in I$ and $q' \in F$

Example language: all words that contain at least two $a \in \Sigma$

# Nominal Automata[2]

A nominal automaton $\mathcal{A}$ over names $\mathbb{A}$ consists of:

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A nominal automaton $\mathcal{A}$ over names $\mathbb{A}$ consists of:

- $\Sigma$, an orbit-finite nominal alphabet
- $Q$, an orbit-finite nominal set of states
- $I \subseteq Q$, an equivariant subset of initial states
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Example language: all words containing their initial letter twice

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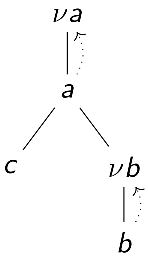

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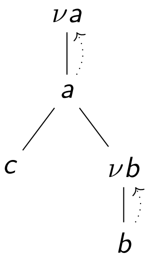


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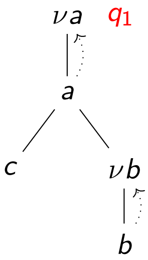


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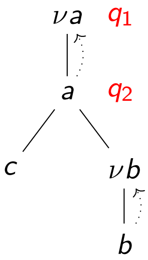


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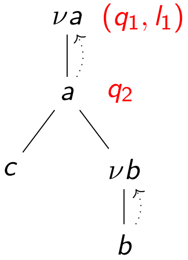


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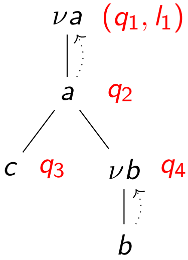


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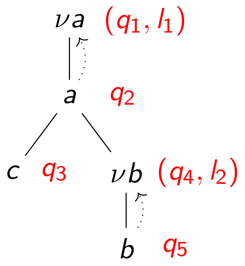


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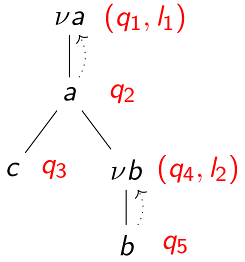


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$\Rightarrow$ closed under union, product + decidable acceptance, emptiness

# Outline

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Type checking, typability, inhabitance all decidable for base types and normal forms!

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- **Relate the work to nominal type theory (Cheney 2009)[9]:**
  Based on nominal set of type variables similar to NNA

# References I

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Moreover, our treatment of $\nu$ is related *name abstraction*[6].