The Blurred Drinker Paradox: Constructive Reverse Mathematics of the Downward Löwenheim-Skolem Theorem

Dominik Kirst
Ben-Gurion University, Israel
Inria Paris, France
dominik.kirst@inria.fr

Haoyi Zeng
Saarland University
Saarbrücken, Germany
haoyi.per@gmail.com

***Abstract*—In the setting of constructive reverse mathematics, we analyse the downward Löwenheim-Skolem (DLS) theorem of first-order logic, stating that every infinite model has a countable elementary submodel. Refining the well-known equivalence of the DLS theorem to the axiom of dependent choice (DC) over classical base theories, our constructive approach allows for several finer logical decompositions: Just assuming countable choice (CC), the DLS theorem is equivalent to the conjunction of DC with a newly identified fragment of the excluded middle (LEM) that we call the blurred drinker paradox (BDP). Further without CC, the DLS theorem is equivalent to the conjunction of BDP with similarly blurred weakenings of DC and CC. Independently of their connection with the DLS theorem, we also study BDP and the blurred choice axioms on their own, for instance by showing that BDP is LEM without a contribution of Markov's principle and that blurred DC is DC without a contribution of CC. The paper is hyperlinked with an accompanying Coq development.**

***Index Terms*—Constructive reverse mathematics, Drinker paradox, Dependent choice, Löwenheim-Skolem theorem, Coq**

I. Introduction

The Löwenheim-Skolem theorem[1] is a central result about first-order logic, entailing that the formalism is incapable of distinguishing different infinite cardinalities. In particular the theorem's so-called downward part, stating that every infinite model (over a countable signature) can be turned into a countably infinite model with otherwise the exact same behaviour, can be considered surprising or even paradoxical:[2] even axiom systems like ZF set theory, concerned with uncountably large sets like the reals or their iterated power sets, admit countable interpretations. This seeming contradiction in particular and its metamathematical relevance in general led to an investigation of the exact logical assumptions under which the downward Löwenheim-Skolem (DLS) theorem applies.

From the perspective of (classical) reverse mathematics [6], [7], there is a definite answer: the DLS theorem is equivalent to the dependent choice axiom (DC), a weak form of the axiom of choice, stating that there is a path through every total relation [8], [9], [10]. To argue the first direction, one can organise the usually iterative construction of the countable submodel such that a single application of DC yields the desired result. For the converse direction, one uses the DLS theorem to turn a given total relation R into a countable subrelation R', applies the classically accepted axiom of countable choice (CC) to obtain a path f' through R', which is then reflected back as a path f through R. In total, that is:

$$\mathsf{DLS} \;\leftrightarrow\; \mathsf{DC}$$

However, the classical answer is insufficient if one investigates the computational content of the DLS theorem, i.e. the question how effective the transformation of a model into a countable submodel really is. A more adequate answer can be obtained by switching to the perspective of *constructive* reverse mathematics [11], [12], which is concerned with the analysis of logical strength over a constructive meta-theory, i.e. in particular without the law of excluded middle (LEM), stating that $p \vee \neg p$ for all propositions p, and ideally also without CC [13]. In that setting, finer logical distinctions become visible and one can analyse the computational content of the DLS theorem by investigating two questions:

1) Does the DLS theorem still follow from DC alone, without any contribution of LEM?
2) Does the DLS theorem still imply the full strength of DC, without any contribution of CC?

$$\mathsf{DLS}\,(+\,\mathsf{CC}) \;\overset{?}{\leftrightarrow}\; \mathsf{DC}\,(+\,\mathsf{LEM})$$

In this paper, after giving a fully constructive proof of a weak form of the DLS theorem sharing the same computational content as constructivised model existence theorems [14], [15], we observe that neither 1) nor 2) is the case. Instead, we clarify which exact fragment of LEM is needed on top of DC to prove the DLS theorem and, conversely, which exact fragment of DC it implies.

Regarding 1), note that the DLS theorem requires LEM in the form of the drinker paradox:[3] in every (non-empty) bar there is a particular person, such that if this person drinks, then everybody in the bar drinks. The classical explanation for that phenomenon is simple, either everyone drinks anyway, in which case we can choose just an arbitrary person, or there is someone not drinking, in which case we choose that person and obtain a contradiction to the assumption they would drink. The role of the drinker paradox in the proof of the

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[1] Usually attributed to Löwenheim [1] and Skolem [2] by name, but credit is also due to Maltsev [3] who in turn credits Tarski.

[2] Discovered and discussed by Skolem [4]. See also the discussion by McCarty and Tennant [5] for a constructivist perspective.

[3] Polularised as a logic puzzle by Smullyan [16] and studied in relation to other principles of constructive mathematics by Escardó and Oliva [17].

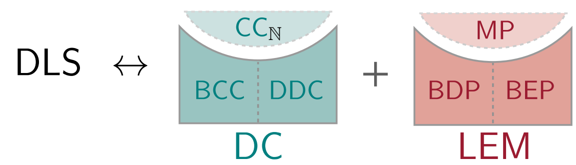
DLS theorem now is to ensure that the constructed model correctly interprets universal quantification:[4] given a formula $\forall x.\, \varphi(x)$ one can find a special domain element a such that $\varphi(a)$ implies $\forall x.\, \varphi(x)$, thereby reducing a test over the whole domain to a test of a single point and easing the correctness proof. However, we observe that one actually does not need to know a concretely but only that it is contained somewhere in the countable model we construct, more formally, that there is a countable subset A such that $\forall a \in A.\, \varphi(a)$ implies $\forall x.\, \varphi(x)$. Seen computationally, this means that we reduce testing over the whole domain to testing only a countable part of it.

On a more abstract level, this observation corresponds to a constructively weaker form of the drinker paradox: in every bar, there is a countable group, such that if everyone in this group drinks, then everybody in the bar drinks. We call this principle the *blurred drinker paradox* as it continues the bar situation at a later point when everyone's vision got blurred and clear identifications of persons become impossible. That it corresponds to the DLS theorem is suggestive since both statements in a sense collapse arbitrary to countable cardinality and indeed we can show that, with CC still assumed in the background, the DLS theorem is equivalent to the conjunction of DC with the blurred drinker paradox. On top of this equivalence, we study the principle (and its dual needed for existential quantification) in a more general setting with arbitrary blurring cardinalities and in relation to other non-constructive principles, unveiling a hierarchy of classically invisible logical structure.

Turning to question 2), we observe that DC becomes underivable from the DLS theorem if we further give up on CC in the background. This suggests that the actual fragment of DC at play is a weakening without the contribution of CC, i.e. a principle that follows from DC but does not imply CC. By a deeper analysis of the proof of the DLS theorem, we actually identify several weakenings of DC that happen to include similar blurring techniques as in the case of the blurred drinker paradox, again connected to the indistinguishability of countable and uncountable cardinalities. In particular, we show that the DLS theorem is equivalent to the conjunction of a strong blurred form of DC and the blurred drinker paradox, with the former further decomposing into a weaker blurred form of DC conjoined with a blurred form of CC.

Orthogonal to its use for the constructive reverse analysis of the DLS theorem, our discussion of blurred choice axioms contributes to the constructive understanding of the logical structure of choice principles in general, thereby complementing related work by Brede and Herbelin [19]. For instance, we show that in the absence of CC, the core of DC actually states that every total relation has a total countable sub-relation or, alternatively, that every directed relation has a directed countable sub-relation. These and similar classically equivalent but constructively weaker reformulations of DC are in visible connection to the DLS theorem.

Our resulting decomposition may then be depicted as



which illustrates that DLS is equivalent to two independent components of DC (abbreviated BCC and DCC) orthogonal to CC on $\mathbb{N}$, in addition to two independent components of LEM (abbreviated BDP and BEP), orthogonal to Markov's principle (MP). Note that the colour-coded abbreviations of all logical principles here and in the remainder of the text are hyperlinked with their definitions in the appendix, where also a more complete diagram of logical connections is given.

While the present paper is written in a deliberately informal way to comply with many systems of (higher-order) constructive mathematics and to address a broad audience, we complement it with a fully mechanised development using the Coq proof assistant [20]. That is, all definitions and theorems have been formalised in the concrete logical foundation underlying Coq such that the correctness of all proofs can be machine-checked. The reasons we do this and actually find it worthwhile are threefold: First, the mechanisation guarantees that all constructions and arguments are correct, which is especially helpful for intricate syntactical arguments needed in the proof of the DLS theorem. Secondly, using a proof assistant actually helped us identify the new non-constructive principles at play by pointing to the constructions and proofs that needed modification. Thirdly, as proving in Coq is programming, the computational content of constructive proofs is made explicit: for instance, the fully constructive proof of the weak DLS theorem, in principle, computes the countable submodel.

Contributions: The contributions of this paper are:

- We introduce the blurred drinker paradox and blurred choice axioms as natural families of logical principles in the context of constructive reverse mathematics. To classify their strength, among others we show that the blurred drinker paradox is LEM without a contribution of Markov's principle (Fact 14) and that the blurred forms of DC are DC without a contribution of CC (Corollary 3).
- Using these logical principles, we give precise constructive decompositions of the DLS theorem: assuming CC, it is equivalent to DC and the blurred drinker paradox (Corollary 2), and without CC, the same equivalence holds for various blurrings of DC and CC (Theorem 5). Moreover, we observe that a weak form of the DLS theorem is fully constructive (Fact 7).
- Our underlying proof strategy for the DLS theorem (Theorem 1) is a streamlining of usual textbook proofs: we construct a syntactic model and collect all structural information in variable environments. Thereby the proof relies neither on signature nor domain extensions and is particularly suitable for computer mechanisation.
- Our paper is accompanied by a Coq development,[5] en-

[4]Incidentally, a similar requirement is needed in Henkin-style completeness proofs [18], connecting to our favoured strategy to establish the DLS theorem.

[5]As part of the Coq-FOL library: https://github.com/uds-psl/coq-library-fol

suring the correctness of all proofs and providing full formal detail, such that the text may remain on a more accessible level. For seamless integration, all definitions and theorems in the PDF version of this paper are hyperlinked with HTML documentation of the code.
- We correct an apparent oversight in the investigation of sub-classical logical principles:[6] in a higher-order logic, the universal closures of the drinker paradox, the existence principle, and the independence of (general) premise are all equivalent to LEM (Fact 2).

Outline: Section II provides an overview of some standard non-constructive axioms and basic concepts of first-order logic. In Section III, we present three constructive versions of the DLS theorem of increasing strength and, in Section IV, we reconstruct the classical equivalence of the DLS theorem to DC. This equivalence is then refined by introducing the blurred drinker paradox in Section V, used in Section VI to replace the use of LEM, and by introducing blurred choice axioms in Section VII, used in Section VIII to replace the use of DC. We close with a discussion concerning the main results, the Coq mechanisation, and future work in Section IX. Note that Sections V and VII are written to be accessible for readers only interested in the new logical principles and their decompositions, independent of their use for the DLS theorem in the other sections.

II. Preliminaries

We work in a constructive meta-theory that we leave underspecified to generalise over concrete standard systems such as intuitionistic higher-order arithmetics like HA^ω, intuitionistic or constructive set theories like IZF and CZF, and constructive type theories like MLTT, HoTT, and CIC. Of course, the latter referring to the Calculus of inductive Constructions [21], [22] implemented in the Coq proof assistant [20] is the concretisation we have in mind, so we also lean towards some type-theoretic notation and jargon.

On the logical level, we stipulate an impredicative collection $\mathbb{P}$ of propositions with standard notation $(\bot, \top, \neg, \wedge, \vee, \forall, \exists)$ to express composite formulas and a means to include inductively defined predicates. On the computational level, we assume collections like $\mathbb{N}$ of natural numbers and $\mathbb{B}$ of Booleans, function spaces like $\mathbb{N}\to\mathbb{B}$, and a means to include inductively defined collections.

We frequently use a *Cantor pairing function* encoding pairs $(n, m) : \mathbb{N}^2$ as numbers $\langle n, m\rangle : \mathbb{N}$. We write $f\,\langle n, m\rangle := \dots$ for function definitions treating an input as an encoded pair.

Given A, if there are functions $i : A\to\mathbb{N}$ and $j : \mathbb{N}\to A$ with $j\,(i\,x) = x$ for all $x : X$, then we say that A is *countable*, where we in particular include finite A to avoid speaking of *at most* countable models in the formulations of the DLS theorem. Note that there are many non-equivalent definitions of countability in constructive logic but for our purposes any of them would do. Similarly, we represent *countable subsets* as functions $f, g : \mathbb{N}\to A$, and write $f \subseteq g$ if for every n there is m with $f\,n = g\,m$ and $f \cup g : \mathbb{N}\to A$ for the subset

$$\begin{aligned}(f \cup g)\,(2n) &:= fn\\ (f \cup g)\,(2n+1) &:= gn\end{aligned}$$

satisfying expectable properties like $f \subseteq f{\cup}g$ and $g \subseteq f{\cup}g$.

Lastly, we call a predicate $P : A\to\mathbb{P}$ *decidable* if it coincides with a Boolean function $f : A\to\mathbb{B}$, i.e. if $\forall x : A.\, P\,x \leftrightarrow f\,x = \mathsf{true}$. This definition naturally generalises to relations $R : A\to B\to\mathbb{P}$.

A. Constructive Reverse Mathematics

The idea of constructive reverse mathematics is to identify non-constructive logical principles and their equivalences to well-known theorems, thereby classifying logical strength and computational content [11], [12], [23]. In preparation of upcoming similar results, we reproduce some well-known connections of logical principles like

$$\begin{aligned}
\mathsf{LEM} &:= \forall p : \mathbb{P}.\, p \vee \neg p\\
\mathsf{LPO} &:= \forall f : \mathbb{N}\to\mathbb{B}.\, (\exists n.\, f\,n = \mathsf{true}) \vee (\forall x.\, f\,n = \mathsf{false})\\
\mathsf{DP}_A &:= \forall P : A\to\mathbb{P}.\, \exists x.\, P\,x \to \forall y.\, P\,y\\
\mathsf{EP}_A &:= \forall P : A\to\mathbb{P}.\, \exists x.\, (\exists y.\, P\,y) \to P\,x\\
\mathsf{IP}_A &:= \forall P : A\to\mathbb{P}.\, \forall p : \mathbb{P}.\, (p \to \exists x.\, P\,x) \to \exists x.\, p \to P\,x
\end{aligned}$$

namely the law of excluded middle, the limited principle of omniscience, the drinker paradox, the existence principle, and the independence of (general) premise principle. In the situation of DP_A for P, we call the given x the *Henkin witness* for P, same for EP_A which is a dual variant of the drinker paradox. We write DP to denote DP_A for all inhabited A, analogously for EP and IP, but state results in the more localised form where possible.

Fact 1. *The following statements hold:*

1. *Both* $\mathsf{DP}_\mathbb{N}$ *and* $\mathsf{EP}_\mathbb{N}$ *imply* LPO.
2. EP_A *is equivalent to* IP_A.

Proof. For (1), assuming $\mathsf{DP}_\mathbb{N}$ and a function $f : \mathbb{N}\to\mathbb{B}$ yields some n such that $f\,n = \mathsf{false}$ implies $f\,n' = \mathsf{false}$ for all n'. Then the claim follows by case analysis of $f\,n$. The claim for $\mathsf{EP}_\mathbb{N}$ follows analogously and (2) is straightforward, with the choice $p := \exists y.\, P\,y$ for the backwards direction. □

In contrast to the situation in first-order logic [24], the universal closures of these principles in a higher-order metatheory with comprehension have the full strength of LEM:

Fact 2. LEM, DP, EP, *and* IP *are equivalent.*

Proof. That LEM implies the other principles is well-known. As an example for the converse, assume DP and some $p : \mathbb{P}$. Using DP for $A := \{b : \mathbb{B} \mid b = \mathsf{false} \vee (p \vee \neg p)\}$ and

$$P\,b := \begin{cases}\neg p & \text{if } b = \mathsf{true}\\ \top & \text{otherwise}\end{cases}$$

yields a Henkin witness $b : A$ for P. If $b = \mathsf{true}$, we directly obtain $p \vee \neg p$ and if $b = \mathsf{false}$, then we derive $\neg p$ as follows:

[6] For instance, a relevant file in the Coq standard library (https://coq.inria.fr/doc/v8.20/stdlib/Coq.Logic.ClassicalFacts.html) refers to both the drinker paradoxes and the independence of premise as principles strictly weaker than LEM, which is actually only the case if one fixes a domain in advance.

On assumption of p we know that true is a member of A and since $P\,b = \top$, by the Henkin property we obtain $P\,b'$ for all $b' : A$. So for $b' :=$ true in A we then obtain $\neg p$, in contradiction to the of assumption p. ☐

While the previous principles concern structure below LEM, we now consider structure below the axiom of choice [25]:

$$\begin{aligned}
\mathsf{AC}_{A,B} &:= \forall R : A{\to}B{\to}\mathbb{P}.\,\mathsf{tot}(R) \to \exists f : A{\to}B.\forall x.\, R\,x\,(f\,x)\\
\mathsf{DC}_{A} &:= \forall R : A{\to}A{\to}\mathbb{P}.\,\mathsf{tot}(R) \to \exists f.\forall n.\, R\,(f\,n)\,(f\,(n+1))\\
\mathsf{CC}_{A} &:= \forall R : \mathbb{N}{\to}A{\to}\mathbb{P}.\,\mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\forall n.\, R\,n\,(f\,n)\\
\mathsf{OAC}_{A,B} &:= \forall R : A{\to}B{\to}\mathbb{P}.\exists f : A{\to}B.\,\mathsf{tot}(R) \to \forall x.\, R\,x\,(f\,x)
\end{aligned}$$

These are the axiom of choice, dependent choice, countable choice, and omniscient choice. Note that the latter is a combination of AC and IP, similar combinations work for other choice axioms:

Fact 3. *For inhabited A and B, $\mathsf{OAC}_{A,B}$ is equivalent to the conjunction of $\mathsf{AC}_{A,B}$ and IP_B.*

Proof. That $\mathsf{OAC}_{A,B}$ implies $\mathsf{AC}_{A,B}$ is obvious and to derive IP_B for $P : B{\to}\mathbb{P}$ one instantiates $\mathsf{OAC}_{A,B}$ to $R\,x\,y := P\,y$. Conversely deriving $\mathsf{OAC}_{A,B}$ for $R : A{\to}B{\to}\mathbb{P}$, note that just using $\mathsf{AC}_{A,B}$ on R would require $\mathsf{IP}_{A\to B}$ to allow postponing the totality proof. Instead, using

$$R'\,x\,y \;:=\; (\exists y'.\, R\,x\,y') \to R\,x\,y$$

we just need IP_B to show R' total to obtain a choice function $f : A{\to}B$ from $\mathsf{AC}_{A,B}$ then also witnessing $\mathsf{OAC}_{A,B}$. ☐

As for the previous principles, we write AC to denote $\mathsf{AC}_{A,B}$ for all A, B and analogously for the other choice principles.

Fact 4. *AC implies DC and DC implies CC.*

Proof. These follow by well-known arguments, see [25] for instance. We sketch the implication from DC to CC to prepare a more general version presented in Fact 17. First note that DC_A can be equivalently stated for arbitrary $x_0 : A$ as

$$\forall R : A{\to}A{\to}\mathbb{P}.\,\mathsf{tot}(R) \to \exists f.\, f\,0 = x_0 \land \forall n.\, R\,(f\,n)\,(f\,(n{+}1))$$

by restricting R to the sub-relation R' reachable from x_0.

Now to show CC, assume a total relation $\mathbb{N}{\to}A{\to}\mathbb{P}$ on A with some element a_0 and consider $A' := \mathbb{N} \times A$ and

$$R'\,(n,x)\,(m,y) \;:=\; m = n+1 \land R\,n\,y$$

which is total since R is total. The modified version of DC for R' and the choice $x_0 := (0, a_0)$ then yields a path $f' : \mathbb{N} \to \mathbb{N} \times A$ through R' and it is straightforward to verify that $f\,n := \pi_2\,(f'\,(n+1))$ is a choice function for R. ☐

We write DC^{Δ} and CC^{Δ} for DC and CC restricted to decidable relations, respectively. We assume that CC^{Δ} holds in our meta-theory, as is the case in most formulations of constructive mathematics, while DC^{Δ} is usually unprovable.

Also, while in (extensional) set-theoretic systems AC implies LEM, this is not the case in most type-theoretic systems, and in neither of those does DC imply LEM.

B. First-Order Logic

We summarise the concepts for first-order logic (FOL) needed to state the downward Löwenheim-Skolem (DLS) theorem. The *syntax* of FOL is represented inductively by terms $t : \mathbb{T}$ and formulas $\varphi : \mathbb{F}$ depending on signatures of function and relation symbols f and P:

$$\begin{aligned}
t : \mathbb{T} \;&::=\; \mathsf{x}_n \mid f\,\vec{t} \qquad\qquad\qquad\qquad (n : \mathbb{N})\\
\varphi, \psi : \mathbb{F} \;&::=\; \dot{\bot} \mid P\,\vec{t} \mid \varphi \dot{\to} \psi \mid \varphi \dot{\land} \psi \mid \varphi \dot{\lor} \psi \mid \dot{\forall}\varphi \mid \dot{\exists}\varphi
\end{aligned}$$

The term vectors $\vec{t}$ are required to have length matching the specified arities $|f|$ and $|P|$ of f and P. The negative fragment of FOL referred to in Facts 6 and 7 comprises formulas only constructed with $\dot{\bot}$, $\dot{\to}$, and $\dot{\forall}$. For the purpose of this paper, we assume that the signatures of function and relation symbols are countable, which induces that so are $\mathbb{T}$ and $\mathbb{F}$.

Variable binding is expressed using de Bruijn indices [26], where a bound variable is encoded as the number of quantifiers shadowing its relevant binder. Capture-avoiding instantiation with parallel substitutions $\sigma : \mathbb{N}{\to}\mathbb{T}$ is defined both for terms as $t[\sigma]$ and formulas as $\varphi[\sigma]$. Notably, $(\dot{\forall}\varphi)[\sigma]$ is defined by $\dot{\forall}[\uparrow \sigma]$ where $\uparrow \sigma$ is a suitable shifting substitution. We denote by $t : \mathbb{T}^c$ and $\varphi : \mathbb{F}^c$ the closed terms and formulas, respectively, i.e. those that do not contain free variables. The latter are also called sentences.

The standard notion of *Tarski semantics* is obtained by interpreting formulas in models $\mathcal{M}$ identified with their underlying domain, providing interpretation functions $\mathcal{M}^{|f|} \to \mathcal{M}$ for each f and relations $\mathcal{M}^{|P|} \to \mathbb{P}$ for each P. Given an environment $\rho : \mathbb{N} \to \mathcal{M}$, we define term evaluation $\hat{\rho}\,t$ and formula satisfaction $\mathcal{M} \vDash_\rho \varphi$ recursively. For instance, the denotation of universal quantifiers is

$$\mathcal{M} \vDash_\rho \dot{\forall}\varphi \;:=\; \forall x : \mathcal{M}.\, \mathcal{M} \vDash_\rho \varphi[x]$$

with $\varphi[x]$ being a notational shorthand expressing that we consider φ in the updated environment mapping the first variable to the domain element x.

While we will mostly be concerned with semantic considerations, to illustrate the connection of the downward Löwenheim-Skolem theorem to completeness, we also briefly use *deduction systems*. Deduction systems are represented by inductive predicates $\Gamma \vdash \varphi$ relating contexts $\Gamma : \mathbb{F}{\to}\mathbb{P}$ with derivable formulas φ, for instance by rules in the style of natural deduction. A classical system is obtained by incorporating a rule like double negation elimination, which in a constructive meta-theory is only sound for classical models, i.e. models satisfying $\mathcal{M} \vDash_\rho \varphi$ or $\mathcal{M} \vDash_\rho \dot{\neg}\varphi$ for all φ.

Fact 5 (Soundness). *If $\Gamma \vdash \varphi$, then $\mathcal{M} \vDash \varphi$ for every classical model $\mathcal{M}$ with $\mathcal{M} \vDash \Gamma$.*

Proof. By induction on the derivation of $\Gamma \vdash \varphi$, most cases are straightforward. To show the classical derivation rule sound, the classicality of the model is required. ☐

The converse property of soundness is completeness, stating that semantic validity implies syntactic provability. In full generality, completeness cannot be proven constructively [27],

[28], [29], [30], [7], but the intermediate model existence theorem is constructive for the negative fragment [14], [15].

Fact 6 (Model Existence). *In the negative fragment of FOL, for every consistent context Γ of sentences one can construct a syntactic model $\mathcal{M}$ over the domain $\mathbb{T}$ such that $\mathcal{M} \vDash \Gamma$.*

Proof. We outline the main construction as it will be relevant for similar syntactic models used in Fact 8 and Theorem 1. In a first step, a constructive version of the Lindenbaum Lemma is used to extend Γ into a consistent context $\Delta \supseteq \Gamma$ with suitable closure properties. Next, a model over domain $\mathbb{T}$ with

$$f^{\mathcal{M}}\,\vec{t} \;:=\; f\,\vec{t} \qquad \text{and} \qquad P^{\mathcal{M}}\,\vec{t} \;:=\; P\,\vec{t} \in \Delta$$

is constructed, for which the so-called Truth Lemma

$$\mathcal{M} \vDash_\sigma \varphi \;\leftrightarrow\; \varphi[\sigma] \in \Delta$$

is verified by induction on φ for all $\sigma : \mathbb{N} \to \mathbb{T}$, acting both as substitution and environment in $\mathcal{M}$. Then since $\Gamma \subseteq \Delta$, in particular $\mathcal{M} \vDash \Gamma$ follows. □

We will see in Fact 7 that the model existence theorem yields a weak but fully constructive formulation of the DLS theorem. This formulation will be based on the notion of elementary equivalence.

Definition 1 (Elementary Equivalence). *Two models $\mathcal{M}$ and $\mathcal{N}$ are* elementarily equivalent *if they satisfy the same sentences, i.e. if for every $\varphi : \mathbb{F}^c$ we have $\mathcal{M} \vDash \varphi$ iff $\mathcal{N} \vDash \varphi$.*

Note that elementarily equivalent models only satisfy the same closed formulas but otherwise may behave extremely differently. A much stronger requirement is that of elementary embeddings, taking all formulas into account and therefore completely aligning the behaviour of the models.

Definition 2 (Elementary Submodel). *Given models $\mathcal{M}$ and $\mathcal{N}$, we call $h : \mathcal{M} \to \mathcal{N}$ an* elementary embedding *if*

$$\forall \rho \varphi.\, \mathcal{M} \vDash_\rho \varphi \leftrightarrow \mathcal{N} \vDash_{h \circ \rho} \varphi.$$

If such an h exists, we call $\mathcal{M}$ an elementary submodel *of $\mathcal{N}$.*

The DLS theorem in full strength then states that every model has a countable elementary submodel.

III. Constructive Löwenheim-Skolem

We begin with a comparison of different constructive proof strategies for the DLS theorem at various strengths, mostly to identify the underlying concepts in preparation of upcoming results. First, a weak formulation only yielding an elementarily equivalent model but not necessarily an elementary submodel is obtained as a by-product of a Henkin-style completeness proof via model existence [18]. Since the Henkin construction is fully constructive in the negative fragment [14], [15], so is the derived DLS theorem.

Fact 7 (DLS via Model Existence). *In the negative fragment of FOL, for every classical model one can construct an elementarily equivalent syntactic model.*

Proof. Given that $\mathcal{M}$ is classical, we can use soundness to show that the set $\mathsf{Th}(\mathcal{M}) := \{\varphi : \mathbb{F}^c \mid \mathcal{M} \vDash \varphi\}$ of closed formulas satisfied by $\mathcal{M}$ is consistent. Then by model existence (Fact 6), there is a model $\mathcal{N}$ with (countable) domain $\mathbb{T}$ and $\mathcal{N} \vDash \mathsf{Th}(\mathcal{M})$. This already establishes the first implication showing $\mathcal{M}$ elementarily equivalent to $\mathcal{N}$. For the converse, assuming a closed formula φ with $\mathcal{N} \vDash \varphi$, we obtain $\mathcal{M} \vDash \varphi$ by using the classicality of $\mathcal{M}$ and the observation that, if it were $\mathcal{M} \vDash \dot\neg\varphi$ instead, also $\mathcal{N} \vDash \dot\neg\varphi$ would follow, contradiction. □

The model existence proof can be extended to the full syntax using LEM alone [15], so the derived version of the DLS theorem notably does not rely on any form of choice axioms. In fact, already the weak law of excluded middle ($\forall p.\, \neg p \vee \neg\neg p$) is sufficient [31] but we are not aware of a proof showing it necessary for this form of the DLS theorem.

Also note that the Lindenbaum extension used in the proof of Fact 6 ensures that quantified formulas have associated Henkin witnesses in form of unused variables. In the second variant, this intermediate step is not necessary, since we restrict to models that address all Henkin witnesses by closed terms.

Definition 3 (Witness Property). *Given a model $\mathcal{M}$ with environment ρ, we call $w : \mathcal{M}$ a* Henkin witness *for $\dot\forall\varphi$ if*

$$\mathcal{M} \vDash_\rho \varphi[w] \;\to\; \mathcal{M} \vDash_\rho \dot\forall\varphi$$

and, symmetrically, a Henkin witness for $\dot\exists\varphi$ if

$$\mathcal{M} \vDash_\rho \dot\exists\varphi \;\to\; \mathcal{M} \vDash_\rho \varphi[w].$$

We say that $\mathcal{M}$ has the witness property *if Henkin witnesses for all formulas can be expressed by closed terms $t : \mathbb{T}^c$.*

For models with the witness property, we can then derive the stronger conclusion yielding a countable elementary submodel by means of a simplified syntactic model construction.

Fact 8 (DLS via Witnesses). *For every model with the witness property one can construct a syntactic elementary submodel.*

Proof. Given $\mathcal{M}$ with the witness property and an arbitrary environment ρ, we consider the syntactic model $\mathcal{N}$ constructed over the (countable) domain $\mathbb{T}$ by setting

$$f^{\mathcal{N}}\,\vec{t} \;:=\; f\,\vec{t} \qquad \text{and} \qquad P^{\mathcal{N}}\,\vec{t} \;:=\; P^{\mathcal{M}}\,(\hat{\rho}\,\vec{t}).$$

We prove that $\hat{\rho}$ is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$, i.e. that $\mathcal{N} \vDash_\sigma \varphi$ if and only if $\mathcal{M} \vDash_{\hat{\rho} \circ \sigma} \varphi$ for all $\sigma : \mathbb{T} \to \mathbb{N}$ and φ by induction on φ. The only cases of interest are the quantifiers, we explain universal quantification as example.

Let $t : \mathbb{T}^c$ denote the Henkin witness for $\dot\forall\varphi$ and assume $\mathcal{N} \vDash_\sigma \dot\forall\varphi$. Then in particular $\mathcal{N} \vDash_\sigma \varphi[t]$ and by inductive hypothesis $\mathcal{M} \vDash_{\hat{\rho} \circ \sigma} \varphi[t]$, which implies $\mathcal{M} \vDash_{\hat{\rho} \circ \sigma} \dot\forall\varphi$ by the Henkin property of t. That conversely $\mathcal{M} \vDash_{\hat{\rho} \circ \sigma} \dot\forall\varphi$ implies $\mathcal{N} \vDash_\sigma \dot\forall\varphi$ is straightforward. □

Many proofs of the DLS theorem proceed by extending the signature with enough fresh constants such that a model satisfying the witness property can be constructed [32]. Alternatively, as a the third variant, we replace the condition to represent Henkin witnesses syntactically with environments collecting them semantically.

Definition 4 (Henkin Environment). *Given a model $\mathcal{M}$, we call $\rho : \mathbb{N}\to\mathcal{M}$ a* Henkin environment *if it collects Henkin witnesses for every formula φ as follows:*

$$\exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\rightarrow\; \mathcal{M} \vDash_\rho \dot\forall\varphi$$
$$\exists n.\, \mathcal{M} \vDash_\rho \dot\exists\varphi \;\rightarrow\; \mathcal{M} \vDash_\rho \varphi[\rho\, n]$$

Note that if $\mathcal{M}$ has the witness property, then $\mathcal{M}$ admits a Henkin environment by enumerating the evaluations of closed terms, but not vice versa.

The use of Henkin environments then allows to conclude the DLS theorem without extending the signature or domain, which is a particularly suitable strategy for mechanisation.

Theorem 1 (DLS via Environments). *For every model admitting a Henkin environment one can construct a syntactic elementary submodel.*

Proof. Given a model $\mathcal{M}$ with Henkin environment ρ, we proceed as in the previous proof, i.e. we consider the syntactic model $\mathcal{N}$ induced by ρ. Again, inductively verifying that $\hat\rho$ is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$ is only non-trivial for quantifiers, for illustration assume $\mathcal{N} \vDash_\sigma \dot\forall\varphi$ for some environment $\sigma : \mathbb{N}\to\mathbb{T}$ and formula φ. We aim to show $\mathcal{M} \vDash_{\hat\rho\circ\sigma} \dot\forall\varphi$ which is equivalent to $\mathcal{M} \vDash_\rho \dot\forall\varphi[\uparrow\sigma]$ and thus reduces to $\mathcal{M} \vDash_\rho \varphi[\uparrow\sigma][\rho\, n]$ using a witness $\rho\, n$ guaranteed by the Henkin property of ρ. The latter then follows from $\mathcal{N} \vDash_\sigma \dot\forall\varphi$ instantiated to $\rho\, n$ and the inductive hypothesis. □

All upcoming proofs of the DLS theorem will factor through Theorem 1 or a strengthening thereof (Theorem 3).

IV. Löwenheim-Skolem using DC and LEM

In this section, we use the proof strategy induced by Theorem 1 to reconstruct the well-known connection of the DLS theorem to DC over a classical meta-theory [9], [10], [8], providing both CC and LEM. First, we show that in this context, DC can be used to construct a Henkin environment and therefore to conclude the DLS theorem. As the later, constructively refined, proofs will follow the same pattern, we give the construction here in full detail.

Theorem 2. *Assuming* DC + LEM*, the* DLS *theorem holds.*

Proof. By Theorem 1, it is enough to show that under the given assumptions every model admits a Henkin environment. Given a model $\mathcal{M}$, the construction of Henkin environment is done in three steps, each making use of a different logical assumption, thereby explaining the respective non-constructive contributions. The high-level idea is to describe an extension method how Henkin witnesses are accumulated stage by stage, where LEM is needed to guarantee the existence of Henkin witnesses, CC (as a consequence of DC) is needed to pick such witnesses simultaneously for every formula in every stage, and finally DC is needed to obtain a path through all stages such that its union constitutes a Henkin environment.

Formally, we express the extension of environments by a step relation $S : (\mathbb{N}\to\mathcal{M})\to(\mathbb{N}\to\mathcal{M})\to\mathbb{P}$ such that $S\,\rho\,\rho'$ captures that ρ' contains all witnesses with respect to ρ:

$$S\,\rho\,\rho' \;:= \rho \subseteq \rho' \wedge \forall\varphi.\, \bigwedge \begin{array}{l} \exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho'\, n] \;\rightarrow\; \mathcal{M} \vDash_\rho \dot\forall\varphi \\ \exists n.\, \mathcal{M} \vDash_\rho \dot\exists\varphi \;\rightarrow\; \mathcal{M} \vDash_\rho \varphi[\rho'\, n] \end{array}$$

Clearly every fixed point of S, i.e. ρ with $S\,\rho\,\rho$, is a Henkin environment so we now explain how such a fixed point is obtained by the aforementioned three steps.

1) Given any environment ρ, the assumption of LEM guarantees Henkin witnesses to exist for all formulas by its connection to the drinker paradoxes: For $\dot\forall\varphi$, the existence of a Henkin witness is exactly the instance $\mathsf{DP}_{\mathcal{M}}$ for the predicate $\mathcal{M} \vDash_\rho \varphi[\_]$ and for $\dot\exists\varphi$ exactly the corresponding instance $\mathsf{EP}_{\mathcal{M}}$.
2) We now use $\mathsf{CC}_{\mathcal{M}}$ to show that S is total, i.e. given some ρ we construct ρ' with $S\,\rho\,\rho'$. By the previous step, we know that every formula $\dot\forall\varphi$ has a Henkin witness with respect to ρ. So by fixing some enumeration φ_n of formulas, we know that for every n the formula $\dot\forall\varphi_n$ has a Henkin witness and thus CC_M yields a function $\rho_\forall$ such that $\rho_\forall\, n$ is the Henkin witness to $\dot\forall\varphi_n$. Analogously, another application of CC_M yields a function $\rho_\exists$ such that $\rho_\exists\, n$ is the Henkin witness to $\dot\exists\varphi_n$. We then set $\rho' := \rho \cup (\rho_\forall \cup \rho_\exists)$ and obtain $S\,\rho\,\rho'$ by simple calculation.
3) We apply $\mathsf{DC}_{\mathbb{N}\to\mathcal{M}}$ to get a path $F : \mathbb{N}\to(\mathbb{N}\to\mathcal{M})$ through S, yielding a cumulative sequence of environments $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$ of Henkin witnesses. To collect the sequence into a single environment, we define
$$\rho\,\langle n_1, n_2\rangle \;:=\; F_{n_1}\, n_2$$
and verify that $S\,\rho\,\rho$, i.e. that ρ is Henkin. This is obtained by composition of several properties of ρ:
   - $F_k \subseteq \rho$ for every k: Given n we need to find n' with $F_k\, n = \rho\, n'$, which holds for $n' := \langle k, n\rangle$.
   - $S\, F_k\, \rho$ for every k: By the previous fact, we know $F_k \subseteq \rho$, so we just need to show that ρ contains all Henkin witnesses relative to F_k. Since F is a path through S, we know $S\, F_k\, F_{k+1}$, so F_{k+1} contains these witnesses, but then so does ρ given $F_{k+1} \subseteq \rho$.
   - $S\,\rho\,\rho$: Since $\rho \subseteq \rho$, we just need to show that for given φ both Henkin witnesses relative to ρ are contained in ρ. Since φ contains only finitely many variables and therefore, since ρ is constructed in cumulative stages, we can find k with $\rho \subseteq_\varphi F_k$, meaning ρ is included in F_k on all free variables of φ. Then in particular there is a permutation substitution σ such that evaluation of φ in ρ coincides with evaluation of $\varphi[\sigma]$ in F_k. But then, since $S\, F_k\, \rho$ by the previous fact, ρ contains the witnesses for $\varphi[\sigma]$ relative to F_k and thus for φ relative to ρ itself. □

We remark that the forthcoming constructive refinements will weaken the respective logical assumptions in each of the three steps above, making precise which independent sources of non-constructivity are at play.

For the converse direction, the necessity for dependent choice relies on the presence of countable choice.

Fact 9. *Assuming* $\mathsf{CC}_{\mathbb{N}}$*, the* DLS *theorem implies* DC.

Proof. The high-level idea is that the DLS theorem reduces DC_A to $\mathsf{CC}_{\mathbb{N}}$ by transforming A into a countable domain.

Formally, assuming a total relation $R : A{\to}A{\to}\mathbb{P}$, we consider the model $\mathcal{M}$ with domain A and interpretation $P_R^{\mathcal{M}}\,x\,y := R\,x\,y$ for some binary relation symbol P_R. The DLS theorem then yields an elementary submodel $\mathcal{N}$ over a countable domain, say $\mathbb{N}$ itself for simplicity, witnessed by an elementary homomorphism $h : \mathcal{N}{\to}\mathcal{M}$. Since totality is a first-order property with $\mathcal{M} \vDash \mathsf{tot}(R)$ by assumption, in particular $\mathcal{N} \vDash \mathsf{tot}(R)$, so the interpretation $P_R^{\mathcal{N}} : \mathbb{N}{\to}\mathbb{N}{\to}\mathbb{P}$ must be total, too.

But then $\mathsf{CC}_{\mathbb{N}}$ yields a choice function $f : \mathbb{N}{\to}\mathbb{N}$ for $P_R^{\mathcal{N}}$ and we can verify that $g : \mathbb{N}{\to}A$ defined by $g\,n := h\,(f^n\,0)$ is a path through R: to justify $R\,(g\,n)\,(g\,(n+1))$ for any n, consider an environment $\rho : \mathbb{N}{\to}\mathcal{N}$ with $\rho\,0 := f^n\,0$ and $\rho\,1 := f^{n+1}\,0$, so $R\,(g\,n)\,(g\,(n{+}1))$ can be equivalently stated as $\mathcal{M} \vDash_{h\circ\rho} P_R(\mathsf{x}_0, x_1)$. By elementarity of h this reduces to $\mathcal{N} \vDash_\rho P_R(\mathsf{x}_0, x_1)$, which translates to $P_R^{\mathcal{N}}\,(f^n\,0)\,(f\,(f^n\,0))$ and holds since f is a choice function for $P_R^{\mathcal{N}}$. □

Corollary 1 (Classical Decomposition). *Over* $\mathsf{CC}_{\mathbb{N}} + \mathsf{LEM}$ *in the background, the* DLS *theorem is equivalent to* DC.

All upcoming derivations of logical principles from the DLS theorem will follow the same pattern of turning a given structure into a countable substructure, deriving a certain property in the simpler countable case, and reflecting it back to the original case. While it seems impossible to derive the full strength of DC from the DLS theorem, as the latter only reduces DC to the constructively still unprovable CC, we observe that the restriction of DC to *decidable* relations can be derived, as it then reduces to the provable principle CC^{Δ}.

Fact 10. *The* DLS *theorem implies* DC^{Δ}.

Proof. As in the proof of Fact 9 we obtain a total relation $P_R^{\mathcal{N}} : \mathbb{N}{\to}\mathbb{N}{\to}\mathbb{P}$ induced by the DLS theorem for a model encoding a total relation $R : A{\to}A{\to}\mathbb{P}$. Now since we assume that R is decidable, so is $P_R^{\mathcal{N}}$ by elementarity and then $\mathsf{CC}_{\mathcal{N}}^{\Delta}$ yields a choice function $f : \mathbb{N}{\to}\mathbb{N}$ for $P_R^{\mathcal{N}}$. From there we proceed as before. □

Regarding the contribution of LEM in the form of the drinker paradoxes needed for the Henkin witnesses in each extension step, there is no chance to fully reverse the result: For instance to derive DP_A, we could start from a predicate $P : A{\to}\mathbb{P}$ but even when using the DLS theorem to reduce P to a countable sub-predicate $P' : \mathbb{N}{\to}\mathbb{P}$, we have no means to find a particular n such that $P'\,n$ would imply $\forall n.\, P'\,n$ and therefore $\forall x.\, P\,x$. In other words, while the DLS theorem reduces DP_A to $\mathsf{DP}_{\mathbb{N}}$, by Fact 1 we would still need at least LPO to proceed deriving $\mathsf{DP}_{\mathbb{N}}$. Instead, in the next section we introduce weakenings of the drinker paradoxes that do become provable in the countable case while still being strong enough to derive the DLS theorem.

V. The Blurred Drinker Paradox

In this section, we introduce the concept of *blurring*, by which we refer to replacing existential quantifiers by quantification over subsets. By this transformation, logical principles can be obtained with constructively slightly reduced information content, as concrete witnesses are hidden in a blur of computationally indistinguishable elements. Here, we study that concept at the example of the drinker paradoxes, in Section VII we will extend it to choice principles. A summary diagram will be given at the end of this section.

We introduce the following blurred forms of DP and EP:

$$\mathsf{BDP}_A^B \;:= \forall P : A{\to}\mathbb{P}. \exists f : B{\to}A.\, (\forall y.\, P\,(f\,y)) \to \forall x.\, P\,x$$
$$\mathsf{BEP}_A^B \;:= \forall P : A{\to}\mathbb{P}. \exists f : B{\to}A.\, (\exists x.\, P\,x) \to \exists y.\, P\,(f\,y)$$

Building on the intuition from before, for instance the principle BDP_A^B states that a Henkin witness for $P : A{\to}\mathbb{P}$ in the sense of DP_A is contained in a blur of size at most B, represented by a function $f : B{\to}A$. In that situation, we call f a *blurred* Henkin witness or simply a *Henkin blur* and require that B is inhabited.

Note that, while DP_A and EP_A are duals in the sense that DP_A also yields EP_A for negative predicates $\{x : A \mid \neg p\,x\}$ and vice versa, even in that sense BEP_A^B is still slightly weaker than BDP_A^B as it concludes with a constructively strong existential quantifier. This will play a role in the slightly different connection to Kripke's schema subject to Fact 13.

We first collect some properties of the introduced principles:

Fact 11. *The following statements hold:*

1. *Both* BDP_A^A *and* BEP_A^A.
2. *If* BDP_A^B *and* BDP_B^C*, then* BDP_A^C.
3. *If* BEP_A^B *and* BEP_B^C*, then* BEP_A^C.
4. DP_A *implies* BDP_A^B *and is equivalent to* $\mathsf{BDP}_A^{\mathbb{1}}$.
5. EP_A *implies* BEP_A^B *and is equivalent to* $\mathsf{BEP}_A^{\mathbb{1}}$.

Proof. We prove each claim independently.

1) By choosing f to be the identity function.
2) Assuming $P : A{\to}\mathbb{P}$, given $f_1 : B{\to}A$ from BDP_A^B for P and $f_2 : C{\to}B$ from BDP_B^C for $P \circ f_1$, the composition $f_1 \circ f_2$ witnesses BDP_A^C for P.
3) Analogous to (2).
4) Assuming $P : A{\to}\mathbb{P}$, DP_A for P yields a Henkin witness x for P and the constant function $f\,y := x$ then witnesses BDP_A^B. Next, if $f : \mathbb{1}{\to}A$ witnesses $\mathsf{BDP}_A^{\mathbb{1}}$ for P, then $f\,\star$ witnesses DP_A for P.
5) Analogous to (4). □

Note that by (1) in particular $\mathsf{BDP}_{\mathbb{N}}^{\mathbb{N}}$ and $\mathsf{BEP}_{\mathbb{N}}^{\mathbb{N}}$ hold, meaning that in light of the concluding remark in Section IV we indeed face weakenings of the drinker paradoxes, provable in the countable case. For simplicity, from now on we write BDP to denote $\mathsf{BDP}_A^{\mathbb{N}}$ for all inhabited A, as the case of countable blurring is the most relevant one, same for BEP.

To illustrate the generality of the blurring concept, we compare the blurred drinker paradox to a blurred form of IP:

$$\mathsf{BIP}_A^B \;:= \forall P : A{\to}\mathbb{P}. \forall p : \mathbb{P}.\, (p \to \exists x.\, P\,x) \to \exists f : B{\to}A.\, p \to \exists y.\, P\,(f\,y)$$

For BIP we could show similar properties as in Fact 11, stating that it is a generalisation of IP into a hierarchy of principles. Instead, we generalise the equivalence of EP and IP recorded in Fact 1.

Fact 12. *BEP_A^B is equivalent to BIP_A^B.*

Proof. Analogous to the proof of Fact 1, for the backwards direction choose $p := \exists x.\, P\, x$ as before. □

Intuitively, the blurred drinker paradoxes allow to test quantified properties on a large domain by considering restrictions to smaller domains, especially countable ones. In this perspective, they resemble Kripke's schema [33], stating that every proposition can be tested by considering the solvability of Boolean functions over countable domain:

$$\mathsf{KS} := \forall p : \mathbb{P}. \exists f : \mathbb{N}\to\mathbb{B}.\, p \leftrightarrow \exists n.\, f\, n = \mathsf{true}$$
$$\mathsf{KS}' := \forall p : \mathbb{P}. \exists f.\, (p \to \neg(\forall n.\, f\, n = \mathsf{false})) \land ((\exists n.\, f\, n = \mathsf{true}) \to p)$$

Note that KS expresses that every proposition is Σ_1, where the logical complexity class Σ_1 refers to the syntactic form of a single existential quantifier over a decidable predicate. In comparison, the slightly weaker KS' replaces the existential quantifier in one direction by a negated universal quantifier.

Fact 13. *BDP implies KS' and BEP implies KS.*

Proof. We show that BEP implies KS, the other claim is similar. So for $p : \mathbb{P}$, consider $A := \{b : \mathbb{B} \mid b = \mathsf{false} \lor p\}$ and

$$P\, b \;:=\; \begin{cases} p & \text{if } b = \mathsf{true} \\ \bot & \text{otherwise} \end{cases}$$

for which $\mathsf{BEP}_A^{\mathbb{N}}$ yields a Henkin blur $f : \mathbb{N}\to A$. The induced underlying function $g : \mathbb{N}\to\mathbb{B}$ then witnesses KS for p: First assuming p, we can show $\exists b.\, P\, b$ by using $b = \mathsf{true}$. Then by the Henkin property of f we obtain $\exists n.\, P\,(f\, n)$ and thus $\exists n.\, g\, n = \mathsf{true}$. Conversely, if $g\, n = \mathsf{true}$ for some n, then by construction p can be derived. □

Note that Kripke's schema can also be formulated for arbitrary B in the role of $\mathbb{N}$, then admitting the same connections for drinker paradoxes blurred by B. In that sense, the latter can be seen as a generalisation of Kripke's schema.

To further characterise the strength of the blurred drinker paradoxes, note that the difference between KS and KS' disappears in the presence of Markov's principle [34], stating that Σ_1 propositions satisfy double negation elimination:

$$\mathsf{MP} \;:=\; \forall f : \mathbb{N}\to\mathbb{B}.\, \neg\neg(\exists n.\, f\, n = \mathsf{true}) \to \exists n.\, f\, n = \mathsf{true}$$

It is straightforward to see that MP follows from LPO and thus from $\mathsf{DP}_{\mathbb{N}}$ by Fact 1. Since it is also well-known that MP together with KS and thus already with KS' implies LEM, we obtain the following decompositions of LEM into blurred drinker paradoxes and side conditions.

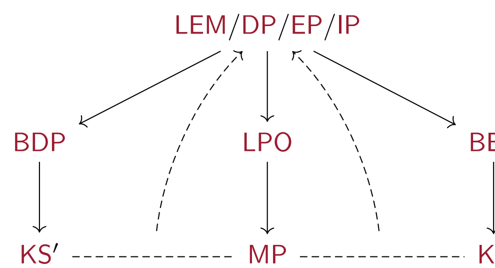
Fact 14. *The following are equivalent to LEM:*

1. $\mathsf{BDP} + \mathsf{DP}_{\mathbb{N}}$
2. $\mathsf{BDP} + \mathsf{MP}$
3. $\mathsf{BEP} + \mathsf{EP}_{\mathbb{N}}$
4. $\mathsf{BEP} + \mathsf{MP}$

Proof. That LEM implies (1)-(4) follows from previous observations. We show that (1) and (4) both imply LEM, analogous arguments work for (2) and (3):

- By Fact 2 it is enough to show DP, i.e. DP_A for every inhabited A. By (1) of Fact 11, this amounts to showing $\mathsf{BDP}_A^{\mathbb{1}}$, which decomposes into $\mathsf{BDP}_A^{\mathbb{N}}$ and $\mathsf{BDP}_{\mathbb{N}}^{\mathbb{1}}$ by (2) of Fact 11. The former is an instance of BDP and the equivalent to $\mathsf{DP}_{\mathbb{N}}$ by again using (1) of Fact 11.
- By Fact 13, BEP implies KS and the latter together with MP implies LEM by a standard argument: Given a proposition p, using KS for the claim $p \lor \neg p$ yields $f : \mathbb{N}\to\mathbb{P}$ such that $p\lor\neg p$ is equivalent to $\exists n.\, f\, n = \mathsf{true}$. By MP, it is enough to show $\neg\neg(\exists n.\, f\, n = \mathsf{true})$ and hence $\neg\neg(p \lor \neg p)$, the latter being a tautology. □

We summarise the connections of the blurred drinker paradoxes with related principles in the following diagram:



Here, the solid arrows depict (presumably strict) implications while the dashed arrows depict combined equivalences.

VI. Löwenheim-Skolem using DC and BDP

We now come back to the DLS theorem and explain how the blurred drinker paradoxes from the previous section capture the contribution of classical logic below LEM, postponing the orthogonal analysis of choice principles below DC. To this end, we first develop a strengthening of Theorem 1 by observing that a weaker form of Henkin environments suffices to construct elementary submodels.

Definition 5 (Blurred Henkin Environment). *Given a model $\mathcal{M}$, we call $\rho : \mathbb{N}\to\mathcal{M}$ a* blurred Henkin environment *if it collects Henkin witnesses for every formula φ as follows:*

$$(\forall n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n]) \;\to\; \mathcal{M} \vDash_\rho \dot\forall\varphi$$
$$\mathcal{M} \vDash_\rho \dot\exists\varphi \;\to\; (\exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n])$$

Note that every Henkin environment is a blurred Henkin environment, but not vice versa. Still, the latter are enough to derive the DLS theorem, as in the construction of the syntactic model actually no concrete witnesses are needed but just a guarantee that they are among the elements selected by the environment.

Theorem 3 (DLS via Blurring). *For every model admitting a blurred Henkin environment one can construct a syntactic elementary submodel.*

Proof. This is basically the same as Theorem 1 where, for instance, in the critical direction of universal quantification we assume that the syntactic model $\mathcal{N}$ induced by ρ satisfies

$\mathcal{N} \vDash_\sigma \dot\forall\varphi$ for some environment $\sigma : \mathbb{T}{\to}\mathbb{N}$ and formula φ and need to show $\mathcal{M} \vDash_{\hat\rho\circ\sigma} \dot\forall\varphi$. The latter is equivalent to $\mathcal{M} \vDash_\rho \dot\forall\varphi[\uparrow\sigma]$ and thus reduces to $\forall n.\, \mathcal{M} \vDash_\rho \varphi[\uparrow\sigma][\rho\, n]$ using the Henkin property of ρ. For some given n, the claim follows from $\mathcal{N} \vDash_\sigma \dot\forall\varphi$ instantiated to $\rho\, n$ and the inductive hypothesis. □

Following the structure of Theorem 2, we now derive the DLS theorem from Theorem 3 by iteratively constructing blurred Henkin environments. The previous use of LEM is now replaced by BDP to accommodate universal quantification, and by BEP to accommodate existential quantification.

Theorem 4. *Over* DC + BDP + BEP *in the background, the* DLS *theorem holds.*

Proof. We employ Theorem 3, leaving us with the construction of a blurred Henkin environment for an arbitrary model $\mathcal{M}$. This construction follows the same outline as in the proof of Theorem 2, i.e. we devise a step relation S accumulating Henkin witnesses and obtain a blurred Henkin environment as a fixed point of S in three steps. As step relation $S\,\rho\,\rho'$, we this time only require that ρ' is a Henkin blur for all formulas φ relative to ρ, instead of the stronger requirement to provide concrete witnesses:

$$S\,\rho\,\rho' \;:=\; \rho \subseteq \rho' \wedge \forall\varphi.\, \bigwedge \begin{array}{l} (\forall n.\, \mathcal{M} \vDash_\rho \varphi[\rho'\, n]) \;\to\; \mathcal{M} \vDash_\rho \dot\forall\varphi \\ \mathcal{M} \vDash_\rho \dot\exists\varphi \;\to\; (\exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho'\, n]) \end{array}$$

1) Given ρ and φ there is a guarantee to proceed, as the instance $\mathsf{BDP}_\mathcal{M}$ for the predicate $\mathcal{M} \vDash_\rho \varphi[\_]$ yields a Henkin blur for $\dot\forall\varphi$ and the same instance of $\mathsf{BEP}_\mathcal{M}$ a Henkin blur for $\dot\exists\varphi$.
2) We derive totality of S at ρ using $\mathsf{CC}_{\mathbb{N}\to\mathcal{M}}$ (following from DC) on the previous fact, thus yielding choice functions $f_\forall, f_\exists : \mathbb{N}{\to}(\mathbb{N}{\to}\mathcal{M})$ such that $f_\forall\, n$ is a Henkin blur for $\dot\forall\varphi_n$ and $f_\exists\, n$ is a Henkin blur for $\dot\exists\varphi_n$. By using Cantor pairing again, they induce environments $\rho_\forall\,\langle n1, n2\rangle := f_\forall\, n_1\, n_2$ and $\rho_\exists\,\langle n1, n2\rangle := f_\exists\, n_1\, n_2$ and for the choice $\rho' := \rho\cup(\rho_\forall\cup\rho_\exists)$ it is straightforward to verify $S\,\rho\,\rho'$ as desired.
3) Finally, we can use $\mathsf{DC}_{\mathbb{N}\to\mathcal{M}}$ to obtain a path $F : \mathbb{N}{\to}(\mathbb{N}{\to}\mathcal{M})$ through S and verify that $\rho\,\langle n1, n2\rangle := F_{n_1}\, n_2$ is a fixed point of S and thus a blurred Henkin environment similarly as before. □

Note that restricting to the negative fragment of FOL, only BDP would be needed, meaning the non-constructive contributions of both sorts of quantification in the DLS theorem are independent. Conversely, from the DLS theorem over the negative fragment we can derive BDP, and with existential quantification present, also BEP becomes derivable.

Fact 15. *The* DLS *theorem implies* BDP + BEP.

Proof. We show how to derive BDP from the DLS theorem, the case of BEP is dual. Similar to the reverse proofs given in Section IV, the high-level idea is that the DLS theorem reduces $\mathsf{BDP}_A^\mathbb{N}$ to the provable $\mathsf{BDP}_\mathbb{N}^\mathbb{N}$.

Formally, assume a predicate $P : A{\to}\mathbb{P}$ for some inhabited A, which we encode as a model $\mathcal{M}$ over A by $P^\mathcal{M}\, x := P\, x$. Then there must be an elementary embedding $h : \mathcal{N}{\to}\mathcal{M}$ from some countable model $\mathcal{N}$, conceived over the domain $\mathbb{N}$ for simplicity.

Since in $\mathcal{N}$ we do have a function $f : \mathbb{N}{\to}\mathbb{N}$ such that $\forall n.\, P^N\,(f\, n)$ implies $\forall n.\, P^\mathcal{N}\, n$, for instance by taking f to be the identity, we obtain that $h \circ f$ is a Henkin blur for P as follows: Assuming $\forall n.\, P\,(h\,(f\, n))$ we show $\forall n.\, P^\mathcal{N}\,(f\, n)$ by fixing n and formulating $P^N\,(f\, n)$ as $\mathcal{N} \vDash_\rho P(\mathsf{x}_0)$ for $\rho\, 0 := f\, n$, which by elementarity follows from $\mathcal{M} \vDash_{h\circ\rho} P(\mathsf{x}_0)$, that is the assumption $P\,(h\,(f\, n))$. But then $\forall x.\, P^\mathcal{N}\, x$, which again reflects up into $\mathcal{M}$ using h and thus yields $\forall x.\, P\, x$. □

Corollary 2 (Blurred Decomposition). *Over* CC *assumed in the background, the* DLS *theorem is equivalent to* DC + BDP + BEP.

That means, disregarding the orthogonal contribution of choice principles, the logical strength of the DLS theorem corresponds exactly to the blurred drinker paradoxes.

VII. Blurred Choice Axioms

In order to complete the analysis, in this section we discuss similarly blurred forms of choice principles that allow a precise decomposition of the DLS theorem. For simplicity, we will consider the concrete case of countable blurring, i.e. using functions $f : \mathbb{N}{\to}A$ but sketch more general formulations at a later point (Section IX-B). Again, a summary diagram will be given at the end of this section.

We begin with a blurring of countable choice that weakens the information provided by a choice function for a total relation by hiding the choices within a countable subset:

$$\mathsf{BCC}_A \;:=\; \forall R : \mathbb{N}{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\forall n.\exists m.\, R\, n\,(f\, m)$$

As usual, we write BCC to denote BCC_A for all A, similarly for all upcoming choice principles. In the situation of BCC_A we call $f : \mathbb{N}{\to}A$ a *blurred* choice function. Note that in the case of $A := \mathbb{N}$ the identity on $\mathbb{N}$ is a blurred choice function, so as in the case of the blurred drinker paradoxes we have the desired property that BCC and all upcoming blurred choice principles hold in the countable case, suggesting their connection to the DLS theorem. Moreover, blurred choice principles follow from their regular counterparts, allowing the following decomposition countable choice:

Fact 16. CC *is equivalent to* $\mathsf{BCC} + \mathsf{CC}_\mathbb{N}$.

Proof. To show that CC_A implies BCC_A, for a total relation $R : \mathbb{N}{\to}A{\to}\mathbb{P}$ we obtain a choice function $f : \mathbb{N}{\to}A$ which in particular can be considered a blurred choice function.

Starting from BCC_A, an application of $\mathsf{CC}_\mathbb{N}$ is enough to turn a blurred choice function into a choice function. □

We will see in Section VIII that BCC is enough to handle step (2) of the construction in Theorem 4, i.e. to derive totality of the step relation S. Regarding step (3), i.e. the derivation of a fixed point for S, we need to find a weakening of DC without the contribution of CC, so that it becomes provable in the countable case. A first attempt is as follows, where we simply replace the path through a total relation R guaranteed by DC by a countable and total sub-relation:

$$\mathsf{BDC}_A \;:=\; \forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{tot}(R \circ f)$$

Note that by $R \circ f$ we refer to the pointwise composition of R and f, i.e. to the relation $R'\, n\, m := R\,(f\, n)\,(f\, m)$. The obtained function f is called a *blurred* path as it still represents a sequence through R but hides the respective continuations.

We then show that, while implying BCC, the obtained BDC needs some contribution of CC to get back the strength of DC.

Fact 17. *The following statements hold:*

1. DC_A *implies* BDC_A.
2. BDC *implies* BCC.
3. DC *is equivalent to* BDC + $\mathsf{CC}_{\mathbb{N}}$.

Proof. We prove all claims independently:

1) Again as in Fact 16, the blurred conclusion of BDC_A is visibly a weakening of the conclusion of DC_A.
2) First as in Fact 4, note that BDC_A can be equivalently stated for arbitrary $x_0 : A$ as

$$\forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f.\, f\, 0 = x_0 \land \mathsf{tot}(R \circ f)$$

by restricting R to the sub-relation R' reachable from x_0. Then a blurred path f through R induces a blurred path f' through R' by first taking the path from x_0 to $f\, 0$ and by then continuing with f.
Now to show BCC, assume a total relation $R : \mathbb{N}{\to}A{\to}\mathbb{P}$ on A with some a_0 and consider $A' := \mathbb{N} \times A$ and

$$R'\,(n,x)\,(m,y) \;:=\; m = n+1 \land R\, n\, y$$

which is total since R is total. The modified version of BDC for R' and the choice $x_0 := (0, a_0)$ then yields a blurred path $f' : \mathbb{N} \to \mathbb{N} \times A$ through R' and it remains to verify that $f\, n := \pi_2\,(f'\, n)$ is a blurred choice function for R.
First, using the properties of f' we derive

$$\forall n.\, \exists m.\, \pi_1(f'\, m) = n$$

by induction on n, choosing 0 in the base case and, in the inductive step where we have some m with $\pi_1(f'\, m) = n$, by choosing m' with $R'\,(f'\, m)\,(f'\, m')$ which we obtain by totality of $R' \circ f'$.
Now, given some n, we find m with $R\, n\,(f\, m)$ by first finding m_1 with $\pi_1(f'\, m_1) = n$ as above and subsequently by finding m_2 with $R'\,(f'\, m_1)\,(f'\, m_2)$ via totality of $R' \circ f'$. Then $R\, n\,(f\, m_2)$ as this is equivalent to $R\,(\pi_1\,(f'\, m_1))\,(\pi_2\,(f'\, m_2))$ which in turn follows from $R'\,(f'\, m_1)\,(f'\, m_2)$.
3) Given (1) and Fact 4 it only remains to show that BDC and $\mathsf{CC}_{\mathbb{N}}$ together imply BDC. So assume some total $R : A{\to}A{\to}\mathbb{P}$, then BDC yields $f : \mathbb{N}{\to}A$ such that $R \circ f$ is total. The latter is a relation $\mathbb{N}{\to}\mathbb{N}{\to}\mathbb{P}$ to which $\mathsf{CC}_{\mathbb{N}}$ yields a choice function $g : \mathbb{N} \to \mathbb{N}$. A path $h : \mathbb{N} \to A$ through R is then obtained by the function $h\, n := f(g^n\, 0)$. ☐

Although BDC therefore yields the desired decomposition of DC, it does not seem strong enough for the purpose regarding the DLS theorem. Intuitively, the problem is that BDC does not have access to the history of previous choices that is needed to merge the environments in proof step (3) of Theorem 4. This problem can be fixed by strengthening to relations on finite sequences A^* or, sufficiently, over pairs A^2:

$$\mathsf{BDC}^2_A \;:=\; \forall R : A^2{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{tot}(R {\circ} f)$$

As for BDC, by $R \circ f$ we refer to pointwise composition of R and f, this time with component wise composition in pairs. First note that BDC^2 is indeed a strengthening of BDC:

Fact 18. BDC^2_A *implies* BDC_A.

Proof. Straightforward by turning $R : A{\to}A{\to}\mathbb{P}$ to show BDC_A into $R'\,(x,y)\,z := R\, x\, z$ and then applying BDC^2_A. ☐

We leave the fact that BDC^2 also corresponds to a version of DC without the contribution of CC to a later point, as this proof will be indirect requiring intermediate structure, see Corollary 3.

As we will see in Section VIII, the principle BDC^2 is already strong enough for the desired purpose regarding replacing DC in the proof of Theorem 4. Moreover, it is possible to again weaken BDC^2 to not even derive BDC, thus completely orthogonalising the different ingredients for the DLS theorem:

$$\mathsf{DDC}_A \;:=\; \forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{dir}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{dir}(R \circ f)$$

Here, by $\mathsf{dir}(R)$ we refer to R being *directed*, i.e. satisfying for every $x, y : A$ that there is $z : A$ with $R\, x\, z$ and $R\, y\, z$. So informally, DDC states that every directed relation as a countable directed sub-relation, which captures the same idea leading to BDC^2 that the information of two previous environments should be combinable.

Indeed, BDC^2 can be decomposed independently into DDC and BCC, with one direction akin to the iterative construction underlying Theorem 4 and the forthcoming Theorem 5.

Fact 19. *The following statements hold:*

1. BDC^2_A *implies* DDC_A.
2. BDC^2 *is equivalent to* DDC + BCC.

Proof. We prove both claims independently:

1) Directedness of $R : A{\to}A{\to}\mathbb{P}$ induces totality of

$$R'\,(x,y)\,y \;:=\; R\, x\, z \land R\, y\, z$$

and, conversely, totality of a countable sub-relation $R' {\circ} f$ induces directedness of $R \circ f$. The claim follows.
2) The first direction follows from (1) and Facts 17 and 18. For the converse, assume a total relation $R : A^2{\to}A{\to}\mathbb{P}$. Consider $S : (\mathbb{N}{\to}A){\to}(\mathbb{N}{\to}A){\to}\mathbb{P}$ defined by

$$S\, \rho\, \rho' \;:=\; \rho \subseteq \rho' \land \forall nm.\exists k.\, R\,(\rho\, m, \rho\, n)\,(\rho'\, k)$$

which can be shown total using BCC as follows: Given some ρ, consider the relation $R' : \mathbb{N}{\to}A$ defined by

$$R'\, \langle n_1, n_2\rangle\, x \;:=\; R\,(\rho\, n_1, \rho\, n_2)\, x$$

which is total since R is total. Then BCC_A yields a blurred choice function $\rho' : \mathbb{N}\to A$ for R' and it is easy to verify that $S\,\rho\,(\rho\cup\rho')$ holds, thus establishing totality of S as desired.

Employing totality, we obtain that S is directed: Given ρ_1 and ρ_2 totality yields ρ_1' and ρ_2' with both $S\,\rho_1\,\rho_1'$ as well as $S\,\rho_2\,\rho_2'$. It then follows that both $S\,\rho_1\,(\rho_1'\cup\rho_2')$ and $S\,\rho_2\,(\rho_1'\cup\rho_2')$ by simple calculation.

We now apply $\mathsf{DDC}_{\mathbb{N}\to X}$ to S and obtain $F : \mathbb{N}\to(\mathbb{N}\to X)$ such that $S\circ F$ is directed. Then $\rho : \mathbb{N}\to X$ defined by

$$\rho\,\langle n_1, n_2\rangle \ := \ F_{n_1}\,n_2$$

can be shown to witness BDC^2 for R as desired: Indeed, to verify that $R\circ\rho$ is total (in fact stating that ρ is a fixed point of S), we assume $n = \langle n_1, n_2\rangle$ and $m = \langle m_1, m_2\rangle$ and need to find k with $R\,(\rho\,n)\,(\rho\,m)\,(\rho\,k)$. Using the directedness of $S\circ F$ for n_1 and m_1, we obtain w with $F_{n_1}\subseteq w$ and $F_{m_1}\subseteq w$, so there are n_3 and m_3 with $F_{n_1}\,n_2 = F_w\,n_3$ and $F_{m_1}\,m_2 = F_w\,m_3$. Moreover, by totality of $S\circ F$ for m we obtain k_1, k_2 with $R\,(F_w\,n_3)\,(F_w\,m_3)\,(F_{k_1}\,k_2)$ and thus $R\,(\rho\,n)\,(\rho\,m)\,(\rho\,k)$ for the choice $k := \langle k_1, k_2\rangle$. □

This decomposition of BDC^2 into DDC and BDC then in particular entails the decomposition of DC into BDC^2 and CC.

Fact 20. DC *implies* BDC^2.

Proof. We first show that DC_A implies a weaker version of DDC_A where the directed relation $R : A\to A\to\mathbb{P}$ is additionally required to be transitive. In that case and since directed relations are total, DC_A yields a path $f : \mathbb{N}\to A$ through R. It then remains to show that $R\circ f$ is directed, which follows since given w.l.o.g. $n < m$ we have both $R\,(f\,n)\,(f(m+1))$ using transitivity of R along the path f connecting n and m, as well as $R\,(f\,m)\,(f(m+1))$ by a single step along f.

Now since the relation F defined in the proof part (2) of Fact 19 is transitive by construction, this modified version of DDC together with BCC, following from DC by Facts 16 and 17, is enough to derive BDC^2 as before. □

Corollary 3. *The following statements hold:*

1. DC *is equivalent to* $\mathsf{BDC}^2 + \mathsf{CC}_{\mathbb{N}}$.
2. DC *is equivalent to* DDC + CC.

Finally we show that, similar to Fact 3, BDC^2 also has an omniscient version that exactly adds BDP and BEP:

$$\mathsf{OBDC}^2_A \ := \ \forall R : A^2\to A\to\mathbb{P}.\exists f : \mathbb{N}\to A.\,\mathsf{tot}(R)\leftrightarrow\mathsf{tot}(R\circ f)$$

We here state only one direction of the decomposition for OBDC^2 as the other direction follows more directly as a by-product of the full analysis of the DLS theorem in Section VIII.

Fact 21. OBDC^2_A *implies* $\mathsf{BDC}^2_A + \mathsf{BDP}_A + \mathsf{BEP}_A$.

Proof. We establish each claim separately:

- That OBDC^2_A implies BDC^2_A is as in Fact 3.
- To derive BDP_A, assume $P : A\to\mathbb{P}$ and set

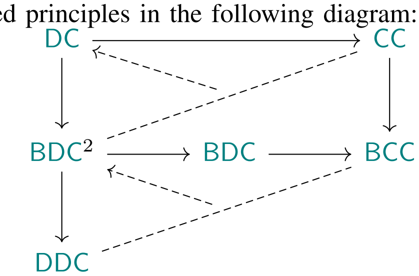
$$R\,(x, y)\,z \ := \ P\,x$$

  for which OBDC^2_A yields $f : \mathbb{N}\to A$ such that R is total if and only if $R\circ f$ is total, reducing to $P\,x$ for all x if and only if $P\,(f\,n)$ for all n. So f also witnesses BDP^2_A.
- To similarly derive BEP_A, assume $P : A\to\mathbb{P}$ and set

$$R\,(x, y)\,z \ := \ P\,z$$

  because then any f such that R is total iff $R\circ f$ is total actually yields $P\,x$ for some x iff $P\,(f\,n)$ for some n. □

We summarise the connections of the blurred choice axioms with related principles in the following diagram:



As with the diagram at the end of Section V, the solid arrows depict (presumably strict) implications while the dashed arrows depict combined equivalences.

VIII. Full Analysis of Löwenheim-Skolem

We conclude the technical part of this paper with the final decomposition of the DLS theorem into the independent logical principles at play and combinations thereof.

Theorem 5 (Decomposition). *The following are equivalent:*

1. *The* DLS *theorem*
2. *The conjunction of* DDC, BCC, BDP, *and* BEP
3. *The conjunction of* BDC^2, BDP, *and* BEP
4. *The principle* OBDC^2

Proof. We establish a circle of implications:

- That (4) implies (3) is by Fact 21.
- That (3) implies (2) is by (2) of Fact 19.
- That (2) implies (1) is a further refinement of Theorem 4. Again using Theorem 3, we demonstrate how a blurred Henkin environment for any model $\mathcal{M}$ can be obtained as a fixed point of the step function S from before:

$$S\,\rho\,\rho' := \rho\subseteq\rho' \land \forall\varphi.\bigwedge \begin{array}{l} (\forall n.\,\mathcal{M}\vDash_\rho \varphi[\rho'\,n]) \ \to\ \mathcal{M}\vDash_\rho \dot\forall\varphi \\ \mathcal{M}\vDash_\rho \dot\exists\varphi \ \to\ (\exists n.\,\mathcal{M}\vDash_\rho \varphi[\rho'\,n]) \end{array}$$

  1) As before, $\mathsf{BDP}_{\mathcal{M}}$ and $\mathsf{BEP}_{\mathcal{M}}$ yield Henkin blurs ρ' for every formula φ and environment ρ.
  2) For totality of S, this time using $\mathsf{BCC}_{\mathbb{N}\to\mathcal{M}}$ instead of $\mathsf{CC}_{\mathbb{N}\to\mathcal{M}}$ yields blurred choice functions $f_\forall, f_\exists : \mathbb{N}\to(\mathbb{N}\to\mathcal{M})$, i.e. we do not have that $f_\forall\,n$ is a Henkin blur for $\dot\forall\varphi_n$ but only know that we can obtain such a Henkin blur by $f_\forall\,m$ for some m. Yet we can still easily verify that for $\rho_\forall$ and $\rho_\exists$ defined by pairing as before and the choice $\rho' := \rho\cup(\rho_\forall\cup\rho_\exists)$ we have that $S\,\rho\,\rho'$.

3) To obtain a fixed point of S using $\mathsf{DDC}_{\mathbb{N}\to\mathcal{M}}$ instead of $\mathsf{DC}_{\mathbb{N}\to\mathcal{M}}$, we first need to argue that S is directed, which given ρ_1 and ρ_2 is easily done by using totality on $\rho_1 \cup \rho_2$. Then from $\mathsf{DDC}_{\mathbb{N}\to\mathcal{M}}$ we obtain $F : \mathbb{N}\to(\mathbb{N}\to\mathcal{M})$ such that $S \circ F$ is directed and verify that the now familiar choice $\rho\,\langle n1, n2\rangle := F_{n_1}\, n_2$ is a fixed point of S and thus a blurred Henkin environment: The proofs that $Fk \subseteq \rho$ and $S\, F_k\, \rho$ are as before and to conclude $S\,\rho\,\rho$, we now use the directedness of $S \circ F$ to show that for every formula φ there is k large enough such that F_k already is a Henkin blur for ρ. For the latter, it is again enough to find k with $\rho \subseteq_\varphi F_k$, which is obtained by directedness for the finitely many F_i contributing to the behaviour of ρ on φ.

- That (1) implies (4) follows the same pattern as all reverse proofs from before, using that $\mathsf{OBDC}^2_{\mathbb{N}}$ is provable. Assuming $R : A^2{\to}A{\to}\mathbb{P}$ on inhabited A taken as model $\mathcal{M}$, from the DLS theorem we obtain an elementary embedding $h : \mathcal{N}{\to}\mathcal{M}$ for a model $\mathcal{N}$ over domain $\mathbb{N}$. For the interpretation $R^{\mathcal{N}}$, e.g. the identity function $f : \mathbb{N}{\to}\mathbb{N}$ satisfies $\mathsf{tot}(R^{\mathcal{N}})$ iff $\mathsf{tot}(R^{\mathcal{N}} \circ f)$. But then by elementarity also $h \circ f$ has that property, i.e. $\mathsf{tot}(R)$ iff $\mathsf{tot}(R^{\mathcal{N}} \circ (h \circ f))$ can be derived as desired. □

Note that all of BDC^2, DDC, and BCC are also directly implied by the DLS theorem, all following the same pattern as the derivation of BDP and BEP already presented in Fact 15.

IX. Discussion

In this paper, we have studied several logical decompositions of the DLS theorem over classical and constructive meta-theories. We briefly summarise the main results as a base for comparison. First, over a fully classical meta-theory, we have:

$$\mathsf{CC}_{\mathbb{N}} + \mathsf{LEM}\ \vdash\ \mathsf{DLS} \leftrightarrow \mathsf{DC} \leftrightarrow \mathsf{BDC}$$

This is the previously known equivalence to DC (Corollary 1), additionally refined by only using BDC as a blurred weakening of DC that is equivalent over $\mathsf{CC}_{\mathbb{N}}$ (Fact 17).

Secondly, assuming just $\mathsf{CC}_{\mathbb{N}}$ in the meta-theory, we obtain:

$$\mathsf{CC}_{\mathbb{N}}\ \vdash\ \mathsf{DLS} \leftrightarrow \mathsf{DC} + \mathsf{BDP} + \mathsf{BEP}$$

This explains which fragment of LEM is needed (Corollary 1), where BDP and BEP independently cover the contribution of syntactic universal and existential quantification. Again, given $\mathsf{CC}_{\mathbb{N}}$ in the background, DC could be replaced by any of its blurrings.

Lastly, in a fully constructive meta-theory, we observe:

$$\vdash\ \mathsf{DLS} \leftrightarrow \mathsf{DDC}{+}\mathsf{BCC}{+}\mathsf{BDP}{+}\mathsf{BEP} \leftrightarrow \mathsf{BDC}^2{+}\mathsf{BDP}{+}\mathsf{BEP}$$

This unveils the individual fragments of DC and CC needed, namely DDC and BCC, which together form BDC^2 (Theorem 5). Using OBDC^2 that integrates BDP and BEP, we finally have:

$$\vdash\ \mathsf{DLS} \leftrightarrow \mathsf{OBDC}^2$$

These decompositions provide a clear logical characterisation of the DLS theorem and the observed principles appear naturally: same as the DLS theorem, they all in one way or another collapse arbitrary to countable cardinality.

A. Coq Mechanisation

The Coq development accompanying this paper is based on and contributed to the Coq library of first-order logic [35]. This library provides the core definitions of syntax, deduction systems, and semantics, as well as a constructive completeness proof we build on for our first approximation of the DLS theorem (Fact 7). The handling of variables is done in the style of the Autosubst 2 framework [36], employing parallel substitutions for de Bruijn indexed syntax and providing a normalisation tactic for substitutive expressions. On top of that library, our development spans roughly 3,500 lines of code, with only around 300 needed for a self-contained proof of the DLS theorem. The latter illustrates that our proof strategy based on variable environments instead of signature or model extension is indeed well-suited for computer mechanisation.

We are aware of a few other mechanisations of the DLS theorem. In Isabelle/HOL, Blanchette and Popescu [37] give a classical and mostly automated proof of the limited strength of our Fact 7, as by-product of a Henkin-style completeness proof. Using Mizar, Caminati [38] also proves the weak form of the DLS theorem corresponding to our Fact 7, again following the strategy factoring through a classical completeness proof. Contained in the Lean mathematical library [39] and contributed by Anderson is a classical proof of the DLS theorem in strong form, i.e. providing an elementary submodel. His proof strategy relies on the full axiom of choice to obtain Skolem functions for arbitrary formulas.

B. Future Work

For the purpose of this paper, we have focused on the case of countable signatures only. As discussed by Espíndola [9] and Karagila [10], the classical equivalence of the DLS theorem to DC generalises to signatures of higher cardinality: for signatures of size A, one needs AC_A on top of DC, which was not visible in the case $A := \mathbb{N}$ since $\mathsf{AC}_{\mathbb{N}}$, that is CC, happens to follow from DC. We conjecture that, in our constructive setting, something similar can be observed, namely that we need the following assumptions: DDC as before, BDP^A and BEP^A now blurred with respect to A, and, in replacement of BCC, a blurred form of the general axiom of choice:

$$\mathsf{BAC}_{A,B}\ :=\ \forall R : A{\to}B{\to}\mathbb{P}.\,\mathsf{tot}(R) \to \exists f : A{\to}B.\forall x.\exists y.\, R\, x\, (f\, y)$$

We have already verified that the DLS theorem at signature size A implies BDP^A, BEP^A, and $\mathsf{BAC}_{A,B}$ for all B if one strengthens the notion of elementary embedding to provide an inverse, but whether they together in turn imply DLS is left for future work. Especially, this proof would require a more conventional proof strategy since our trick to use variable environments, with $\mathbb{N}$ as domain, to represent submodels, now with A as domain, is certainly not applicable.

Another interesting direction is to consider the upwards case of the Löwenheim-Skolem theorem, stating that every infinite model has an elementary extension at arbitrarily larger cardinality. The proof usually employs the compactness theorem to ensure the distinctness of newly added elements to increase the cardinality. The compactness theorem, however, is known to not be constructive, leaving the constructive status of the the upwards Löwenheim-Skolem theorem to be investigated.

Finally, our working hypothesis regarding the status of the blurred logical principles is that neither of them collapses, i.e. that BDP + BEP does not imply LEM, that BCC does not imply CC, that DDC does not imply BCC, and that BDC does not imply BDC^2. Recall that most implications do not hold locally, i.e. it is not the case that for instance BCC_A always implies CC_A, as $\mathsf{BCC}_\mathbb{N}$ is provable while $\mathsf{CC}_\mathbb{N}$ is not. Therefore, only the question of global implications remains of interest and to obtain full certainty, one has to construct separating models. A promising approach is the use of realisability models, where the logical components are interpreted by computational means. For instance, non-deterministic realisability allows to invalidate CC (and thus also DC) [40] and there is hope that in this setting still BCC (and maybe even BDC) are validated. Moreover, based on classical realisability, Castro [41] independently identifies and separates an instance of BDC he calls collection axiom. Finally, realisability models incorporating quotation operations [42] might be useful to separate BDP and BEP from LEM.

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Appendix

a) Overview of Logical Principles: Standard principles below the excluded middle:

$$\begin{aligned}
\mathsf{LEM} &:= \forall p : \mathbb{P}.\, p \lor \neg p \\
\mathsf{LPO} &:= \forall f : \mathbb{N}{\to}\mathbb{B}.\, (\exists n.\, f\,n = \mathsf{true}) \lor (\forall x.\, f\,n = \mathsf{false}) \\
\mathsf{DP}_A &:= \forall P : A{\to}\mathbb{P}.\, \exists x.\, P\,x \to \forall y.\, P\,y \\
\mathsf{EP}_A &:= \forall P : A{\to}\mathbb{P}.\, \exists x.\, (\exists y.\, P\,y) \to P\,x \\
\mathsf{IP}_A &:= \forall P : A{\to}\mathbb{P}.\, \forall p : \mathbb{P}.\, (p \to \exists x.\, P\,x) \to \exists x.\, p \to P\,x \\
\mathsf{KS} &:= \forall p : \mathbb{P}. \exists f : \mathbb{N}{\to}\mathbb{B}.\, p \leftrightarrow \exists n.\, f\,n = \mathsf{true} \\
\mathsf{MP} &:= \forall f : \mathbb{N}{\to}\mathbb{B}.\, \neg\neg(\exists n.\, f\,n = \mathsf{true}) \to \exists n.\, f\,n = \mathsf{true}
\end{aligned}$$

Standard principles below the axiom of choice:

$$\begin{aligned}
\mathsf{AC}_{A,B} &:= \forall R : A{\to}B{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : A{\to}B. \forall x.\, R\,x\,(f\,x) \\
\mathsf{DC}_A &:= \forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A. \forall n.\, R\,(f\,n)\,(f\,(n+1)) \\
\mathsf{CC}_A &:= \forall R : \mathbb{N}{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A. \forall n.\, R\,n\,(f\,n) \\
\mathsf{OAC}_{A,B} &:= \forall R : A{\to}B{\to}\mathbb{P}. \exists f : A{\to}B.\, \mathsf{tot}(R) \to \forall x.\, R\,x\,(f\,x)
\end{aligned}$$

Blurred principles below the excluded middle:

$$\begin{aligned}
\mathsf{BDP}_A^B &:= \forall P : A{\to}\mathbb{P}. \exists f : B{\to}A.\, (\forall y.\, P\,(f\,y)) \to \forall x.\, P\,x \\
\mathsf{BEP}_A^B &:= \forall P : A{\to}\mathbb{P}. \exists f : B{\to}A.\, (\exists x.\, P\,x) \to \exists y.\, P\,(f\,y) \\
\mathsf{BIP}_A^B &:= \forall P : A{\to}\mathbb{P}.\, \forall p : \mathbb{P}.\, (p \to \exists x.\, P\,x) \\
&\quad \to \exists f : B{\to}A.\, p \to \exists y.\, P\,(f\,y)
\end{aligned}$$

Blurred principles below the axiom of choice:

$$\begin{aligned}
\mathsf{BCC}_A &:= \forall R : \mathbb{N}{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A. \forall n. \exists m.\, R\,n\,(f\,m) \\
\mathsf{BDC}_A &:= \forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{tot}(R \circ f) \\
\mathsf{BDC}_A^2 &:= \forall R : A^2{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{tot}(R \circ f) \\
\mathsf{DDC}_A &:= \forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{dir}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{dir}(R \circ f) \\
\mathsf{OBDC}_A^2 &:= \forall R : A^2{\to}A{\to}\mathbb{P}. \exists f : \mathbb{N}{\to}A.\, \mathsf{tot}(R) \leftrightarrow \mathsf{tot}(R \circ f)
\end{aligned}$$

b) Connections of Logical Principles: See below for an overview of our main results regarding DLS. As before, solid arrows depict (presumably strict) implications while the dashed arrows depict combined equivalences. Moreover, double arrows depict direct equivalences with potential side conditions placed next to the arrows.