Functions, Ordinals and Well-Orderings in Coq

FINAL BACHELOR SEMINAR TALK

DOMINIK KIRST

The Aim

- Substantial development of Set Theory in Coq
- Partial Formalisation of Hrbacek & Jech 1999:
  1. Basic Set Theory
  2. Functions
  3. Ordinals
  4. Hartogs Numbers
  5. Transfinite Recursion
  6. Well-Ordering Theorem
- First part of thesis: explaination of proofs

The Aim ctd.

- Comparison of two and a half proofs:
  - Zermelo 1904
  - Zermelo 1908
  - Textbook proof (Hrbacek & Jech)
- Discussion of the Axiom of Choice
- Brief summary of related work:
  - Ilik (Zermelo 1904 in Agda)
  - Brown (Zermelo 1908 in Coq)
  - Kaiser (Zermelo 1904 in Coq and set-theoretic model)

The Status (1)

Sets as types coming with an ∈-relation:

```
Parameter set: Type.
Parameter el: set -> set -> Prop.
```

We assume ZF and define usual set-operations:

```
Parameter pair: set -> set -> set.
Axiom Pair: forall A B x, x ∈ (pair A B) <-> x = A ∨ x = B.
```

$$\Downarrow$$

```
Definition opair A B := pair (pair A A) (pair A B).
Definition relation R A B := R c= product A B.
```

The Status (2)

Object Functions as Propositions:

```
Definition function f A B :=
    relation f A B ∧ total f A B ∧ functional f
```

Helpful application operator:

```
Lemma app_cor f A B x:
    function f A B -> x ∈ A -> (x,f[x]) ∈ f
```

⇒ Powerful and convenient framework

The Status (3)

Von Neumann definition of ordinals:

1. Set-Transitive
2. Wellordered by Element-Ordering

Important Result of this section:

```
Theorem ordinal_wo M:
    (exists a, ordinal a ∧ iso M a) -> WO M
```

The Status (4)

Construction of Ordertypes yields:

```
Lemma wo_ordiso M R:
    wordering R M -> (exists! a, ordiso M a R)

Theorem wo_ordinal M:
    WO M -> (exists a, ordinal a ∧ iso M a R)
```

Defining Hartogs Numbers allows:

```
Theorem hartogs A:
    forall A', A' c= A -> ~ iso (h A) A'
```

The Status (5)

We assume Bounded Transfinite Recursion (BTR):

- Advantage: gives proper functions as sets
- Enough for our purpose
- Formulation as in Hrbacek & Jech, Theorem 4.4

```
Theorem trans_rec a B g:
  ordinal a -> function g (space a B) B ->
  ∃! f, function f a B ∧ ∀ a'∈ a, f[a'] = g[f|a']
```

⇒ Proof can be given as possible add-on

The Status (6)

Proof of Well-Ordering Theorem:

- For set M find ordinal λ such that $\lambda \cong M$
- Using BTR, AC and Hartogs Numbers

Proof of Equivalence:

- Given WO for arbitrary sets we construct AC
- Enables further equivalence proofs (Zorn etc.)

Appendix

Are the orderings in Zer. 1904 and Zer. 1908 equal?

- Akihiro Kanomori (2004): Yes! But no proof…
- Idea:

1. $O1$: set of all γ-sets
2. $O2$: intersection of all Θ-chains
3. It turns out, that $S \in O1$ iff $M \setminus S \in O2$
4. This justifies $x <_1 y$ iff $x <_2 y$

$\Rightarrow$ Formalisation could be future work

Thanks for your attention!

References:

- Ernst Zermelo 1904:
  „Beweis, daß jede Menge wohlgeordnet werden kann.“
- Ernst Zermelo 1908:
  „Neuer Beweis für die Möglichkeit einer Wohlordnung.“
- Akihiro Kanamori 2004:
  „Zermelo and Set Theory“
- Danko Ilik 2007:
  „Zermelo's Well-Ordering Theorem in Type Theory
- Karel Hrbacek & Thomas Jech 1999:
  „Introduction to Set Theory“

Website: https://www.ps.uni-saarland.de/~kirst